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Quantum matrix algebra for the $SU(n)$ WZNW model

P Furlan[1,2], L K Hadjiivanov[2,3], A P Isaev[4], O V Ogievetsky[5,6], P N Pyatov[4] and I T Todorov[3]

[1] Dipartimento di Fisica Teorica dell' Università degli Studi di Trieste, Strada Costiera 11, I-34014 Trieste, Italy
[2] Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, Trieste, Italy
[3] Theoretical Physics Division, Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria
[4] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, 141 980 Moscow Region, Russia
[5] Centre de Physique Théorique, Luminy, F-13288 Marseille, France

E-mail: furlan@trieste.infn.it, lhadji@inrne.bas.bg, isaevap@thsun1.jinr.ru, oleg@cpt.univ-mrs.fr, pyatov@thsun1.jinr.ru and todorov@inrne.bas.bg

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Abstract
The zero modes of the chiral $SU(n)$ WZNW model give rise to an *intertwining quantum matrix algebra* $\mathcal{A}$ generated by an $n \times n$ matrix $a = \left(a_\alpha^i\right), i, \alpha = 1, \ldots, n$ (with noncommuting entries) and by rational functions of n commuting elements q^{p_i} satisfying $\prod_{i=1}^n q^{p_i} = 1, q^{p_i} a_\alpha^j = a_\alpha^j q^{p_i + \delta_i^j - \frac{1}{n}}$. We study a generalization of the Fock space ($\mathcal{F}$) representation of $\mathcal{A}$ for generic q (q not a root of unity) and demonstrate that it gives rise to a *model* of the quantum universal enveloping algebra $U_q = U_q(sl_n)$, with each irreducible representation entering $\mathcal{F}$ with multiplicity 1. For an integer $\widehat{su}(n)$ *height* $h\,(= k + n \geqslant n)$ the complex parameter q is an even root of unity, $q^h = -1$, and the algebra $\mathcal{A}$ has an ideal $\mathcal{I}_h$ such that the factor algebra $\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h$ is finite dimensional. All physical U_q modules—of shifted weights satisfying $p_{1n} \equiv p_1 - p_n < h$—appear in the Fock representation of $\mathcal{A}_h$.

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Introduction

Although the Wess–Zumino–Novikov–Witten (WZNW) model was first formulated in terms of a (multivalued) action [65], it was originally solved [52] by using axiomatic conformal

[6] On leave of absence from: P N Lebedev Physical Institute, Theoretical Department, 117924 Moscow, Leninsky prospect 53, Russia.

field theory methods. The two-dimensional (2D) Euclidean Green functions have been expressed [10] as sums of products of analytic and antianalytic *conformal blocks*. Their operator interpretation exhibits some puzzling features: the presence of *noninteger* ('quantum') *statistical dimensions* (that appear as positive real solutions of the *fusion rules* [64]) contrasted with the local ('Bose') commutation relations (CR) of the corresponding 2D fields. The gradual understanding of both the factorization property and the hidden *braid group statistics* (signalled by the quantum dimensions) only begins with the development of the canonical approach to the model (for a sample of references, see [6, 8, 27, 28, 30, 34–36, 38, 40]) and the associated splitting of the basic group valued field $g : \mathbb{S}^1 \times \mathbb{R} \to G$ into chiral parts. The resulting zero mode extended phase space displays a new type of *quantum* group *gauge symmetry*: on the one hand, it is expressed in terms of the *quantum universal enveloping algebra* $U_q(\mathcal{G})$, a deformation of the *finite-dimensional Lie algebra* $\mathcal{G}$ of G—much like a gauge symmetry of the first kind; on the other, it requires the introduction of an *extended, indefinite metric state space*, a typical feature of a (local) gauge theory of the second kind.

Chiral fields admit an expansion into *chiral vertex operators* (CVO) [63] which diagonalize the monodromy and are expressed in terms of the currents' degrees of freedom with 'zero mode' coefficients that are independent of the world sheet coordinate [2, 15, 35, 36, 38]. Such a type of quantum theory has been studied in the framework of lattice current algebras (see [3, 16, 28, 30, 40] and references therein). Its accurate formulation in the continuum limit has only been attempted in the case of $G = SU(2)$ (see [26, 35, 36]). The identification (in [45]) of the zero mode (U_q) vertex operators a^i_α (the 'U_q oscillators' of the $SU(2)$ case [35]) with the generators of a *quantum matrix algebra* defined by a pair of (dynamical) R-matrices allows us to extend this approach to the case of $G = SU(n)$.

The basic group valued chiral field $u^A_\alpha(x)$ is thus expanded in CVO $u^A_i(x, p)$ which interpolate between chiral current algebra modules of weight $p = p_j v^{(j)}$ and $p + v^{(i)}, i = 1, \ldots, n$ (in the notation of [45] to be recapitulated in section 1). The operator valued coefficients a^i_α of the resulting expansion intertwine finite-dimensional irreducible representations (IR) of $U_q \equiv U_q(sl_n)$ that are labelled by the same weights. For generic q (q not a root of unity) they generate, acting on a suitably defined vacuum vector, a Fock-like space $\mathcal{F}$ that contains every (finite-dimensional) IR of U_q with multiplicity 1, thus providing a *model* for U_q in the sense of [11]. This result (established in section 3.1) appears to be novel even in the undeformed case ($q = 1$) giving rise to a new (for $n > 2$) model of $SU(n)$. In the important case of q an even root of unity ($q^h = -1$) we have prepared the ground (in sections 3.2 and 3.3) for a (co)homological study of the 2D (left and right movers') zero mode problem [26].

It should be emphasized that displaying the quantum group's degrees of freedom requires an extension of the phase space of the models under consideration. Much interesting work on both physical and mathematical aspects of 2D conformal field theory has been performed without going to such an extension—see e.g. [7, 10, 31, 52, 55]. The concept of a quantum group, on the other hand, has emerged in the study of closely related integrable systems and its uncovering in conformal field theory models has fascinated researchers from the outset—see e.g. [6, 27, 56, 57]. (For a historical survey of an early stage of this development see [43]. Significant later developments in different directions—beyond the scope of the present paper—can be found, e.g., in [13, 32, 54, 58].)

Even within the scope of this paper there remain unresolved problems. We have, for instance, no operator realization of the extended chiral WZNW model, involving indecomposable highest weight modules of the Kac–Moody current algebra.

The paper is organized as follows. Section 1 provides an updated summary of recent work [34–36] on the $SU(n)$ WZNW model. A new point here is the accurate treatment of the path dependence of the exchange relations in both the x and the $z = e^{ix}$ pictures (proposition 1.3). In section 2 we carry out the factorization of the chiral field $u(x)$ into CVO and U_q vertex operators and review relevant results of [45] computing, in particular, the determinant of the quantum matrix a as a function of the $U_q(sl_n)$ weights. The discussion of the interrelation between the braiding properties of 4-point blocks and the exchange relations among zero modes presented in section 2.2 is new; so are some technical results like proposition 2.3 used in what follows. Section 3.1 introduces the Fock space ($\mathcal{F}$) representation of the zero mode algebra $\mathcal{A}$ for generic q; the main result is summed up in proposition 3.3. In section 3.2 we compute inner products for the canonical bases in the U_q modules $\mathcal{F}_p$ for $n = 2, 3$. In section 3.3 we study the kernel of the inner product in $\mathcal{F}$ for q an even root of unity,

$$q = e^{-i\frac{\pi}{h}} \qquad (h = k + n \geqslant n). \tag{0.1}$$

It is presented in the form $\tilde{\mathcal{I}}_h\mathcal{F}$ where $\tilde{\mathcal{I}}_h$ is an ideal in $\mathcal{A}$. We select a smaller ideal $\mathcal{I}_h \subset \tilde{\mathcal{I}}_h$ (introduced in [45]) such that the factor algebra $\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h$ is still finite dimensional but contains along with each physical weight p (with $p_{1n} < h$) a weight $\tilde{p}$ corresponding to the first singular vector of the associated Kac–Moody module (cf remark 2.1).

1. Monodromy extended $SU(n)$ WZNW model: a synopsis

1.1. Exchange relations; path dependent monomials of chiral fields

The WZNW action for a group valued field on a cylindric spacetime $\mathbb{R}^1 \times \mathbb{S}^1$ is written as

$$S = -\frac{k}{4\pi}\int\{\mathrm{Tr}(g^{-1}\partial_+ g)(g^{-1}\partial_- g)\,dx^+\,dx^- + s^*\omega(g)\} \qquad x^\pm = x \pm t \tag{1.1a}$$

where $s^*\omega$ is the pullback ($s^*g^{-1}\,dg = g^{-1}\partial_+ g\,dx^+ + g^{-1}\partial_- g\,dx^-$) of a 2-form ω on G satisfying

$$d\omega(g) = \tfrac{1}{3}\mathrm{Tr}(g^{-1}\,dg)^3. \tag{1.1b}$$

The general, $G = SU(n)$ valued (periodic) solution, $g(t, x + 2\pi) = g(t, x)$, of the resulting equations of motion factorizes into a product of group valued chiral fields

$$g^A_B(t, x) = u^A_\alpha(x + t)(\bar{u}^{-1})^\alpha_B(x - t) \qquad (\text{classically, } g, u, \bar{u} \in SU(n)) \tag{1.2a}$$

where u and $\bar{u}$ satisfy a twisted periodicity condition

$$u(x + 2\pi) = u(x)M \qquad \bar{u}(x + 2\pi) = \bar{u}(x)\bar{M} \tag{1.2b}$$

with *equal monodromies*, $\bar{M} = M$. The symplectic form of the 2D model is expressed as a sum of two chiral 2-forms involving the monodromy:

$$\begin{aligned}\Omega^{(2)} &= \Omega(u, M) - \Omega(\bar{u}, M)\\ \Omega(u, M) &= \frac{k}{4\pi}\left(\mathrm{Tr}\left(\int_{-\pi}^{\pi}\partial(u^{-1}\,du)u^{-1}\,du\,dx - b^{-1}\,db\,dMM^{-1}\right) + \rho(M)\right).\end{aligned} \tag{1.3}$$

Here $b = u(-\pi)$ and the 2-form $\rho(M)$ is restricted by the requirement that $\Omega(u, M)$ is closed, $d\Omega(u, M) = 0$ which is equivalent to

$$d\rho(M) = \tfrac{1}{3}\,\mathrm{Tr}(dMM^{-1})^3 \tag{1.4a}$$

(in other words, ρ satisfies the same equation (1.1b) as ω).

Such a ρ can only be defined locally—in an open dense neighbourhood of the identity of the complexification of $SU(n)$ to $SL(n, \mathbb{C})$. An example is given by

$$\rho(M) = \mathrm{Tr}\big(M_+^{-1}\,dM_+M_-^{-1}\,dM_-\big) \tag{1.4b}$$

where $M_\pm$ are the Gauss components of M (which are well defined for $M_{nn} \neq 0 \neq \det\begin{pmatrix} M_{n-1n-1} & M_{n-1n} \\ M_{nn-1} & M_{nn}\end{pmatrix}$ etc),

$$M = M_+ M_-^{-1} \qquad M_+ = N_+ D \qquad M_-^{-1} = N_- D \tag{1.5}$$

$$N_+ = \begin{pmatrix} 1 & f_1 & f_{12} & \dots \\ 0 & 1 & f_2 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \qquad N_- = \begin{pmatrix} 1 & 0 & 0 & \dots \\ e_1 & 1 & 0 & \dots \\ e_{21} & e_2 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \qquad D = \left(d_\alpha \delta^\alpha_\beta\right) \tag{1.6}$$

and the common diagonal matrix D has unit determinant: $d_1 d_2 \cdots d_n = 1$. Different solutions ρ of (1.4*a*) correspond to different non-degenerate solutions of the classical Yang–Baxter equation [34, 40].

The closed 2-form (1.3) on the space of chiral variables $u, \bar{u}, M$ is degenerate. This fact is related to the non-uniqueness of the decomposition (1.2*a*): $g(t, x)$ does not change under constant right shifts of the chiral components, $u \to uh, \bar{u} \to \bar{u}h, h \in G$. Under such shifts the monodromy changes as $M \to h^{-1}Mh$ (see also the discussion of this point in [9]). We restore non-degeneracy by further extending the phase space, assuming that the monodromies M and $\bar{M}$ of u and $\bar{u}$ are independent so that the left and the right sectors completely decouple. As a result, monodromy invariance in the extended phase space is lost since M and $\bar{M}$ satisfy Poisson bracket relations of opposite sign (due to (1.3)) and hence cannot be identified. Single-valuedness of $g(t, x)$ can only be recovered in a weak sense, when g is applied to a suitable subspace of 'physical states' in the quantum theory [25, 26, 35, 36].

We require that quantization respects all symmetries of the classical chiral theory. Apart from conformal invariance and invariance under periodic left shifts the (u, M) system admits a Poisson–Lie symmetry under constant right shifts [4, 34, 60] which gives rise to a quantum group symmetry in the quantized theory. The quantum exchange relations so obtained [27, 28, 30, 34–36, 40],

$$u_2(y)u_1(x) = u_1(x)u_2(y)R(x-y) \qquad \bar{u}_1(x)\bar{u}_2(y) = \bar{u}_2(y)\bar{u}_1(x)R(x-y) \tag{1.7}$$

(for $0 < |x - y| < 2\pi$) can also be written as braid relations:

$$P u_1(y) u_2(x) = u_1(x) u_2(y) \hat{R}(x-y) \tag{1.7a}$$

$$\bar{u}_1^{-1}(y)\bar{u}_2^{-1}(x)P = \hat{R}^{-1}(x-y)\bar{u}_1^{-1}(x)\bar{u}_2^{-1}(y) \quad\Leftrightarrow\quad \bar{u}_1(x)\bar{u}_2(y)P = \bar{u}_2(y)\bar{u}_1(x)\hat{R}(x-y). \tag{1.7b}$$

Here $R(x)$ is related to the (constant, Jimbo) $SL(n)R$-matrix [50] by

$$R(x) = R\theta(x) + PR^{-1}P\theta(-x) \tag{1.8a}$$

$$R^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \bar{q}^{\frac{1}{n}}\left(\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} q^{\delta_{\alpha_1\alpha_2}} + (q - \bar{q})\delta^{\alpha_1\alpha_2}_{\beta_2\beta_1}\theta_{\alpha_1\alpha_2}\right) \tag{1.8b}$$

where

$$\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} \qquad \theta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha \leqslant \beta \end{cases} \qquad \bar{q} := q^{-1} \tag{1.8c}$$

P stands for permutation of factors in $V \otimes V$, $V = \mathbb{C}^n$, while $\hat{R}$ is the corresponding braid operator:

$$\hat{R} = RP \qquad P(X_{|1\rangle}Y_{|2\rangle}) = X_{|2\rangle}Y_{|1\rangle} \tag{1.9a}$$

$$\hat{R}(x) = R(x)P = \begin{cases} \hat{R} & \text{for} \quad x > 0 \\ \hat{R}^{-1} & \text{for} \quad x < 0. \end{cases} \tag{1.9b}$$

We use throughout the tensor product notation of Faddeev *et al* [29]: $u_1 = u \otimes \mathbf{I}$, $u_2 = \mathbf{I} \otimes u$ are thus defined as operators in $V \otimes V$.

Restoring all indices we can write equation (1.7*a*) as

$$u^B_\alpha(y) u^A_\beta(x) = u^A_\sigma(x) u^B_\tau(y) \hat{R}(x-y)^{\sigma\tau}_{\alpha\beta}. \tag{1.7c}$$

Whenever dealing with a tensor product of three or more copies of V we shall write R_{ij} to indicate that R acts non-trivially on the ith and jth factors (and reduces to the identity operator on all others).

Remark 1.1. The operator $\hat{R}$ (1.9*a*) coincides with $\hat{R}_{21} = P\hat{R}_{12}P$ ($P = P_{12}$) in the notation of [29] and [45]. We note that if $\hat{R}_{ii+1}$ satisfy the Artin braid relations then so do $\hat{R}_{i+1i}$; we have, in particular,

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \quad \Leftrightarrow \quad \hat{R}_{32}\hat{R}_{21}\hat{R}_{32} = \hat{R}_{21}\hat{R}_{32}\hat{R}_{21}. \tag{1.10}$$

Indeed, the two relations are obtained from one another by acting from left and right on both sides with the permutation operator $P_{13} = P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23}$ ($=P_{31}$) and taking into account the identities

$$P_{13}\hat{R}_{12}P_{13} = \hat{R}_{32} \qquad P_{13}\hat{R}_{23}P_{13} = \hat{R}_{21}. \tag{1.11}$$

Here we shall stick, following [34–36], to the form (1.7), (1.9) of the basic exchange relations. Note however that (1.1*a*) involves a change of sign in the WZ term (as compared to [34–36]) which yields the exchange of the x^+ and x^- factors in (1.2*a*) and is responsible for the sign change in the phase of q (0.1).

The multivaluedness of chiral fields requires a more precise formulation of (1.7). To give an unambiguous meaning to such exchange relations we shall proceed as follows.

Energy positivity implies that for any $l > 0$ the vector valued function

$$\Psi(\zeta_1, \ldots, \zeta_l) = u_1(\zeta_1) \cdots u_l(\zeta_l)|0\rangle$$

is (single valued) analytic on a simply connected open subset

$$\{\zeta_j = x_j + \mathrm{i}y_j;\ |x_j| < \pi,\ j = 1, \ldots, l;\ y_j < y_{j+1},\ j = 1, \ldots, l-1\}$$

($x_{jk} := x_j - x_k$) of the manifold $\mathbb{C}^l \backslash \mathrm{Diag}$ where Diag is defined as the partial diagonal set in $\mathbb{C}^l$: $\mathrm{Diag} = \{(\zeta_1, \ldots, \zeta_l),\ \zeta_j = \zeta_k \text{ for some } j \neq k\}$.

Introduce (exploiting reparametrization invariance—cf [39]) the *analytic* (z-) *picture* fundamental chiral field

$$\varphi(z) = \mathrm{e}^{-\mathrm{i}\Delta\zeta} u(\zeta) \qquad z = \mathrm{e}^{\mathrm{i}\zeta} \qquad \Delta = \frac{n^2-1}{2hn} \tag{1.12}$$

Δ standing for the conformal dimension of u, and note that the variables z_j are radially ordered in the domain $\mathcal{O}_l$:

$$\mathcal{O}_l = \{z_j = \mathrm{e}^{-y_j + \mathrm{i}x_j};\ |z_j| > |z_{j+1}|,\ j = 1, \ldots, l-1;\ |\arg z_j| < \pi,\ j = 1, \ldots, l\}. \tag{1.13}$$

Remark 1.2. The *time evolution* law

$$\mathrm{e}^{\mathrm{i}tL_0} u(x)\, \mathrm{e}^{-\mathrm{i}tL_0} = u(x+t) \tag{1.14a}$$

for the *'real compact picture' field* $u(x)$ implies that

$$\mathrm{e}^{\mathrm{i}tL_0} \varphi(z)\, \mathrm{e}^{-\mathrm{i}tL_0} = \mathrm{e}^{\mathrm{i}t\Delta} \varphi(z\, \mathrm{e}^{\mathrm{i}t}). \tag{1.14b}$$

Energy positivity, combined with the pre-factor in (1.12), guarantees that the state vector $\varphi(z)|0\rangle$ is a single valued analytic function of z in the neighbourhood of the origin (in fact, for a suitably defined inner product, its Taylor expansion around $z = 0$ is norm convergent for $|z| < 1$—see [22]).

The vector valued functions

$$\Phi(z_1, \ldots, z_l) = \varphi_1(z_1) \cdots \varphi_l(z_l)|0\rangle \tag{1.15a}$$

and

$$\Psi(\zeta_1, \ldots, \zeta_l) = u_1(\zeta_1) \cdots u_l(\zeta_l)|0\rangle = \prod_j \mathrm{e}^{\mathrm{i}\Delta\zeta_j}\,\Phi(\mathrm{e}^{\mathrm{i}\zeta_1}, \ldots, \mathrm{e}^{\mathrm{i}\zeta_l}) \tag{1.15b}$$

are both analytic in their respective domains (cf (1.13)) and are real analytic (and still single valued) on the parts

$$\{\zeta_j = x_j (\Rightarrow z_j = \mathrm{e}^{\mathrm{i}x_j}), x_1 > x_2 > \cdots > x_l, x_{1l} < \pi\}$$

of their physical boundaries.

The following proposition allows us to continue these boundary values through the domain $\mathcal{O}_l$ to any other ordered set of x_j (the result will be a path dependent multivalued function for $\{z_1, \ldots, z_l\} \in \mathbb{C}^l \backslash \mathrm{Diag}$).

Proposition 1.3. *Let* $z_1 = \mathrm{e}^{\mathrm{i}x_1}$, $z_2 = \mathrm{e}^{\mathrm{i}x_2}$, $0 < x_{12} < 2\pi$; *the path exchanging* x_1 *and* x_2 *(and hence* z_1 *and* z_2*),*

$$C_{12} : \zeta_{1,2}(t) = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}t}\left(x_{1,2}\cos\frac{\pi}{2}t + \mathrm{i}x_{2,1}\sin\frac{\pi}{2}t\right) \qquad 0 \leqslant t \leqslant 1 \tag{1.16a}$$

turns clockwise around the middle of the segment (x_1, x_2):

$$\zeta_1(t) + \zeta_2(t) = x_1 + x_2 \qquad \zeta_{12}(t) := \zeta_1(t) - \zeta_2(t) = x_{12}\,\mathrm{e}^{-\mathrm{i}\pi t}. \tag{1.16b}$$

Furthermore, if $z_a(t) = \mathrm{e}^{\mathrm{i}\zeta_a(t)}$, $a = 1, 2$, *then*

$$|z_1(t)|^2 = \mathrm{e}^{x_{12}\sin\pi t} = |z_2(t)|^{-2} > 1 \qquad \text{for} \quad 0 < t < 1 \tag{1.16c}$$

so that the pair $(z_1(t), z_2(t))$ *satisfies the requirement (1.13) for two consecutive arguments in the analyticity domain* $\mathcal{O}_l$. *For* $0 < x_{21} < 2\pi$ *one has to change the sign of t (and thus the orientation of the path (1.16)) in order to preserve the inequality* $|z_1(t)| > |z_2(t)|$.

Proof. All assertions are verified by a direct computation; in particular, (1.16*b*) implies that

$$2\,\mathrm{Im}\,\zeta_2(t) = x_{12}\sin\pi t = -2\,\mathrm{Im}\,\zeta_1(t) \tag{1.16d}$$

which yields (1.16*c*). □

We note that for $\zeta_{1,2}$ given by (1.16*a*) one has $|\zeta_1(t)|^2 + |\zeta_2(t)|^2 = x_1^2 + x_2^2$. Proposition 1.3 supplements (1.12) in describing the relationship (the essential equivalence) between the real compact and the analytic picture allowing us to use each time the one better adapted to the problem under consideration.

We are now prepared to give an unambiguous formulation of the exchange relations (1.7).

Let $\Pi_{12}u_1(x_2)u_2(x_1)$ $(\Pi_{12}\varphi_1(z_2)\varphi_2(z_1))$ be the analytic continuation of $u_1(x_1)u_2(x_2)$ (respectively $\varphi_1(z_1)\varphi_2(z_2)$) along a path in the homotopy class of C_{12} (1.16). Then equation (1.7*a*) should be substituted by

$$\begin{aligned} P\Pi_{12}u_1(x_2)u_2(x_1) &= u_1(x_1)u_2(x_2)\hat{R} \\ P\Pi_{12}\varphi_1(z_2)\varphi_2(z_1) &= \varphi_1(z_1)\varphi_2(z_2)\hat{R} \end{aligned} \tag{1.7d}$$

for $z_j = e^{ix_j}$, $0 < x_{12} < 2\pi$. For $0 < x_{21} < 2\pi$ and a positively oriented path one should replace $\hat{R}$ by $\hat{R}^{-1}$.

We recall (see [36]) that the quantized u (and g) cannot be treated as group elements. We can just assert that the operator product expansion of u with its conjugate only involves fields of the family (or, rather, the Verma module) of the unit operator. The relation

$$u(x + 2\pi) = e^{2\pi i L_0} u(x)\, e^{-2\pi i L_0} = u(x) M \tag{1.17}$$

on the other hand, gives (by (1.14) for Δ given by (1.12))

$$\left(M^{\alpha}_{\beta} - q^{\frac{1}{n}-n}\delta^{\alpha}_{\beta}\right)|0\rangle = 0; \tag{1.18a}$$

hence, in order to preserve the condition $d_1 \cdots d_n = 1$ for the product of diagonal elements of M_+ and M_-^{-1} we should substitute (1.5) by its quantum version

$$M = q^{\frac{1}{n}-n} M_+ M_-^{-1}. \tag{1.18b}$$

The tensor products of Gauss components, $M_{2\pm} M_{1\pm}$, of the monodromy matrix commute with the braid operator,

$$[\hat{R}, M_{2\pm} M_{1\pm}] = 0 = [\hat{R}, \bar{M}_{1\pm} \bar{M}_{2\pm}] \tag{1.19a}$$

(and hence, with its inverse) but

$$\hat{R} M_{2-} M_{1+} = M_{2+} M_{1-} \hat{R} \qquad \hat{R} \bar{M}_{1+} \bar{M}_{2-} = \bar{M}_{1-} \bar{M}_{2+} \hat{R} \tag{1.19b}$$

while the exchange relations between u and $M_\pm$ can be written in the form (cf [34–36])

$$M_{1\pm} P u_1(x) = u_2(x) \hat{R}^{\mp 1} M_{2\pm} \qquad \bar{M}_{2\pm} P \bar{u}_2(x) = \bar{u}_1(x) \hat{R}^{\pm} \bar{M}_{1\pm}. \tag{1.20}$$

The left and right sectors decouple completely as a consequence of the separation of variables in the classical extended phase space,

$$[M_1, \bar{u}_2] = [u_1, \bar{u}_2] = [M_1, \bar{M}_2] = [u_1, \bar{M}_2] = 0. \tag{1.21}$$

The above relations for the left sector variables (u, M) are invariant under the left coaction of $SL_q(n)$,

$$u^A_\alpha(x) \to (T^{-1})^{\beta}_{\alpha} \otimes u^A_\beta \equiv (u^A(x) T^{-1})_\alpha \qquad M^\alpha_\beta \to T^\alpha_\gamma (T^{-1})^\delta_\beta \otimes M^\gamma_\delta \equiv (T M T^{-1})^\alpha_\beta \tag{1.22a}$$

while the right sector is invariant under its right coaction,

$$\bar{u}^\alpha_A(x) \to \bar{u}^\beta_A(x) \otimes (\bar{T})^\alpha_\beta = (\bar{T} \bar{u}_A(x))^\alpha \qquad \bar{M}^\alpha_\beta \to \bar{M}^\gamma_\delta \otimes \bar{T}^\alpha_\gamma (\bar{T}^{-1})^\delta_\beta \equiv (\bar{T} \bar{M} \bar{T}^{-1})^\alpha_\beta \tag{1.22b}$$

provided

$$\hat{R} T_2 T_1 = T_2 T_1 \hat{R} \qquad \hat{R} \bar{T}_1 \bar{T}_2 = \bar{T}_1 \bar{T}_2 \hat{R} \tag{1.23}$$

where we have used concise notation on the right-hand side of (1.22). The elements $T^\alpha{}_\beta$ of T commute with $u, M, \bar{u}$ and $\bar{M}$. The fact that the maps (1.22a) and (1.22b) are respectively left and right coactions [45] can be proved by checking the comodule axioms, see e.g. [1, 48]. There are corresponding transformations of the elements of $M_\pm$ and $\bar{M}_\pm$.

Thus the Latin and Greek indices of u and $\bar{u}$ in (1.2a) transform differently: A, B correspond to the (undeformed) $SU(n)$ action while α is a quantum group index.

It is known, on the other hand, that the first equations in (1.19a) and (1.19b) for the matrices $M_\pm$ are equivalent to the defining relations of the ('simply connected' [21]) quantum universal enveloping algebra (QUEA) $U_q(sl_n)$ that is paired by duality to Fun$(SL_q(n))$

(see [29]). The Chevalley generators of U_q are related to the elements d_i, e_i, f_i of the matrices (1.5) by ([29]; see also [36])

$$\begin{aligned} &d_i = q^{\Lambda_{i-1}-\Lambda_i} && (i = 1, \ldots, n, \Lambda_0 = 0 = \Lambda_n) \\ &e_i = (\bar{q} - q)E_i && f_i = (\bar{q} - q)F_i \end{aligned} \tag{1.24a}$$

$$\begin{aligned} &(\bar{q} - q)f_{12} = f_2 f_1 - q f_1 f_2 = (\bar{q} - q)^2(F_2F_1 - qF_1F_2) \qquad \text{etc} \\ &(\bar{q} - q)e_{21} = e_1e_2 - qe_2e_1 = (\bar{q} - q)^2(E_1E_2 - qE_2E_1) \qquad \text{etc.} \end{aligned} \tag{1.24b}$$

Here Λ_i are the fundamental co-weights of $sl(n)$ (related to the co-roots H_i by $H_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$); E_i and F_i are the raising and lowering operators satisfying

$$\begin{aligned} &[E_i, F_j] = [H_i]\delta_{ij} && \left([H] := \frac{q^H - \bar{q}^H}{q - \bar{q}}\right) \\ &[E_i, E_j] = 0 = [F_i, F_j] && \text{for} \quad |j - i| \geqslant 2 \end{aligned} \tag{1.25a}$$

$$\begin{aligned} &q^{\Lambda_i}E_j = E_j q^{\Lambda_i+\delta_{ij}} && q^{\Lambda_i}F_j = F_j q^{\Lambda_i-\delta_{ij}} \\ &[2]\, X_i X_{i\pm1} X_i = X_{i\pm1}X_i^2 + X_i^2 X_{i\pm1} && \text{for} \quad X = E, F. \end{aligned} \tag{1.25b}$$

We note that the invariance under the coaction of $SL_q(n)$ (1.22a) is, in effect, equivalent to the covariance relations

$$\begin{aligned} &q^{H_i}u_\alpha(x)\bar{q}^{H_i} = q^{\delta^i_\alpha - \delta^{i+1}_\alpha}u_\alpha(x) \qquad [E_i, u_\alpha] = \delta^{i+1}_\alpha u_{\alpha-1}(x)q^{H_i} \\ &F_i u_\alpha(x) - q^{\delta^{i+1}_\alpha - \delta^i_\alpha}u_\alpha(x)F_i = \delta^i_\alpha u_{\alpha+1}(x). \end{aligned} \tag{1.26}$$

1.2. R-matrix realizations of the Hecke algebra; quantum antisymmetrizers

The R-matrix for the quantum deformation of any (simple) Lie algebra can be obtained as a representation of Drinfeld's universal R-matrix [23]. In the case of the defining representation of $SU(n)$ the braid operator (1.9) gives rise, in addition, to a representation of the *Hecke algebra*. This fact, exploited in [45], is important for our understanding of the dynamical R-matrix. We recall the basic definitions.

For any integer $k \geqslant 2$ let $H_k(q)$ be an associative algebra with generators $1, g_1, \ldots, g_{k-1}$, depending on a non-zero complex parameter q, with defining relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for} \quad 1 \leqslant i \leqslant k-2 \quad (\text{if } k \geqslant 3) \tag{1.27a}$$

$$g_i g_j = g_j g_i \qquad \text{for} \quad |i - j| \neq 1 \quad 1 \leqslant i, j \leqslant k-1 \tag{1.27b}$$

$$g_i^2 = 1 + (q - \bar{q})g_i \qquad \text{for} \quad 1 \leqslant i \leqslant k-1 \quad \bar{q} := q^{-1}. \tag{1.27c}$$

The $SL(n)$ braid operator $\hat{R}$ (see (1.8b), (1.8c) and (1.9a)) generates a representation $\rho_n : H_k(q) \to \text{End}(V^{\otimes k})$, $V = \mathbb{C}^n$ for any $k \geqslant 2$,

$$\rho_n(g_i) = q^{\frac{1}{n}}\hat{R}_{ii+1} \quad \text{or} \quad [\rho_n(g_i)]^{\pm1} = q^{\pm1}\mathbf{I} - A_i \tag{1.28a}$$

where A is the q-antisymmetrizer

$$A^{\alpha_1\alpha_2}_{\beta_1\beta_2} = q^{\epsilon_{\alpha_2\alpha_1}}\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} - \delta^{\alpha_1\alpha_2}_{\beta_2\beta_1} \qquad q^{\epsilon_{\alpha_2\alpha_1}} = \begin{cases} \bar{q} & \text{for} \quad \alpha_1 > \alpha_2 \\ 1 & \text{for} \quad \alpha_1 = \alpha_2 \\ q & \text{for} \quad \alpha_1 < \alpha_2 \end{cases} \tag{1.28b}$$

($A_i = [2]A^{(i+1,i)}$ in the—suitably extended—notation of [45]; note that for $q^2 = -1$, $[2] = 0$ the 'normalized antisymmetrizer' $A^{(i+1,i)}$ is ill-defined while A_i still makes sense).

Equations (1.27) are equivalent to the following relations for the antisymmetrizers A_i:

$$A_i A_{i+1} A_i - A_i = A_{i+1} A_i A_{i+1} - A_{i+1} \tag{1.29a}$$

$$A_i A_j = A_j A_i \qquad \text{for} \quad |i-j| \neq 1 \tag{1.29b}$$

$$A_i^2 = [2] A_i. \tag{1.29c}$$

Remark 1.3. We can define (see, e.g., [44]) the higher antisymmetrizers $A_{ij}, i < j$ inductively, setting

$$\begin{aligned} A_{ij+1} := {} & A_{ij}(q^{j-i+1} - q^{j-i}\rho_n(g_j) + \cdots + (-1)^{j-i+1}\rho_n(g_j g_{j-1} \cdots g_i)) \\ = {} & A_{i+1j+1}(q^{j-i+1} - q^{j-i}\rho_n(g_i) + \cdots + (-1)^{j-i+1}\rho_n(g_i g_{i+1} \cdots g_j)). \end{aligned} \tag{1.30a}$$

They can also be expressed in terms of antisymmetrizers only:

$$\begin{aligned} A_{ii+1} &= A_i \\ A_{ij+1} &= \frac{1}{[j-i]!}(A_{ij} A_j A_{ij} - [j-i][j-i]! A_{ij}) \\ &= \frac{1}{[j-i]!}(A_{i+1j+1} A_i A_{i+1j+1} - [j-i][j-i]! A_{i+1j+1}). \end{aligned} \tag{1.30b}$$

The term 'q-antisymmetrizer' is justified by the relation

$$(\rho_n(g_i) + \bar{q}) A_{1j} = 0 = A_{1j}(\rho_n(g_i) + \bar{q}) \tag{1.31a}$$

or

$$A_{1i} A_{1j} = [i]! A_{1j} \qquad \text{for} \quad 1 < i \leqslant j. \tag{1.31b}$$

The dependence of the representation ρ_n on n (for $G = SU(n)$) is manifest in the relations

$$A_{1n+1} = 0 \qquad \operatorname{rank} A_{1n} = 1. \tag{1.32}$$

A_{1n} can be written as a (tensor) product of two Levi-Civita tensors,

$$A_{1n} = \mathcal{E}^{|1\ldots n\rangle} \mathcal{E}_{\langle 1\ldots n|} \qquad \mathcal{E}_{\langle 1\ldots n|} \mathcal{E}^{|1\ldots n\rangle} = [n]! \tag{1.33}$$

the second equation implying summation in all n repeated indices. We can (and shall) choose the covariant and the contravariant $\mathcal{E}$-tensors equal,

$$\mathcal{E}^{\alpha_1\alpha_2\ldots\alpha_n} = \mathcal{E}_{\alpha_1\alpha_2\ldots\alpha_n} = \bar{q}^{n(n-1)/4}(-q)^{\ell(\sigma)} \qquad \text{for} \quad \sigma = \begin{pmatrix} n, \ldots, 1 \\ \alpha_1, \ldots, \alpha_n \end{pmatrix} (\in \mathcal{S}_n); \tag{1.34}$$

here $\ell(\sigma)$ is the length of the permutation σ. (Note the difference between (1.34) and expression (2.5) of [45] for $\mathcal{E}$ which can be traced back to our present choice (1.28) for $\rho_n(g_i)$—our $\hat{R}_{ii+1}$ corresponding to $\hat{R}_{i+1i}$ of [45]—cf remark 1.1.)

We recall for further reference that the first equation (1.32) is equivalent to either of the following two relations,

$$A_{1n} A_{2n+1} A_{1n} = ([n-1]!)^2 A_{1n} \tag{1.35a}$$

$$A_{2n+1} A_{1n} A_{2n+1} = ([n-1]!)^2 A_{2n+1} \tag{1.35b}$$

(see lemma 1.1 of [45]); this agrees with (1.33) and (1.34) since

$$\mathcal{E}_{\langle 2\ldots n+1|}\mathcal{E}^{|1\ldots n\rangle} = (-1)^{n-1}[n-1]!\delta^{|1\rangle}_{\langle n+1|} \tag{1.36a}$$

$$\mathcal{E}_{\langle 1\ldots n|}\mathcal{E}^{|2\ldots n+1\rangle} = (-1)^{n-1}[n-1]!\delta^{|n+1\rangle}_{\langle 1|}. \tag{1.36b}$$

We shall encounter in section 2 another, 'dynamical', Hecke algebraic representation of the braid group which has the same form (1.28*a*) but with a 'dynamical antisymmetrizer', i.e. $A_i = A_i(p)$, a (rational) function of the q-weights $(q^{p_1}, \ldots, q^{p_n})$ which satisfies a finite difference ('dynamical') version of (1.29*a*).

1.3. Barycentric basis, shifted $su(n)$ weights; conformal dimensions

Let $\{v^{(i)}, i = 1, \ldots, n\}$ be a symmetric '*barycentric basis*' of (linearly dependent) real traceless diagonal matrices (thus $\{v^{(j)}\}$ span a real Cartan subalgebra $\mathfrak{h} \subset sl(n)$):

$$(v^{(i)})^j_k = \left(\delta_{ij} - \frac{1}{n}\right)\delta^j_k \quad\Rightarrow\quad \sum_{i=1}^n v^{(i)} = 0 \qquad (v^{(i)}|v^{(j)}) = \delta_{ij} - \frac{1}{n}. \tag{1.37}$$

(The inner product of two matrices is given by the trace of their product.) Analogously, the n 'barycentric' components p_i of a vector in the $(n-1)$-dimensional dual space $\mathfrak{h}^*$ are determined up to a common additive constant and can be fixed by requiring $\sum_{i=1}^n p_i = 0$. Specifying thus the bases, we can make any such vector in $\mathfrak{h}^*$ correspond to a unique diagonal matrix $p \in \mathfrak{h}$,

$$p = \sum_{i=1}^n p_i v^{(i)} = \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & p_n \end{pmatrix} \qquad \sum_{i=1}^n p_i = 0 \tag{1.38}$$

and vice versa. In particular, the simple $sl(n)$ roots α_i and the fundamental $sl(n)$ weights $\Lambda^{(j)}, i, j = 1, \ldots, n-1$, satisfying $(\Lambda^{(j)}|\alpha_i) = \delta^j_i$, correspond to the following diagonal matrices (denoted by the same symbols),

$$\alpha_i = v^{(i)} - v^{(i+1)} \qquad \Lambda^{(j)} = \sum_{\ell=1}^j v^{(\ell)} \equiv \left(1 - \frac{j}{n}\right)\sum_{\ell=1}^j v^{(\ell)} - \frac{j}{n}\sum_{\ell=j+1}^n v^{(\ell)} \tag{1.39}$$

respectively. Expanding p (1.38) in the basis of fundamental weights,

$$p = \sum_{i=1}^n p_i v^{(i)} = \sum_{j=1}^{n-1} p_{jj+1}\Lambda^{(j)} \qquad p_{ij} := p_i - p_j$$

one can characterize a *shifted dominant weight*

$$p = \Lambda + \rho \qquad \Lambda = \sum_{i=1}^{n-1}\lambda_i\Lambda^{(i)} \qquad \lambda_i \in \mathbb{Z}_+ \qquad \rho = \sum_{i=1}^{n-1}\Lambda^{(i)} = \frac{1}{2}\sum_{\alpha>0}\alpha \tag{1.40}$$

(ρ is the $sl(n)$ Weyl vector) by the relations

$$p_{ii+1} = \lambda_i + 1 \in \mathbb{N} \qquad i = 1, 2, \ldots, n-1. \tag{1.41}$$

The non-negative integers $\lambda_i = p_{ii+1} - 1$ count the number of columns of length i in the Young tableau that corresponds to the IR of highest weight p of $SU(n)$—see, e.g., [33]. Conversely, p_i satisfying (1.38) can be expressed in terms of the integer valued differences p_{ij} as $p_i = \frac{1}{n}\sum_{j=1}^n p_{ij}$.

Dominant weights p also label highest weight representations of U_q. For integer heights h ($\geqslant n$) and q satisfying (0.1) these are (unitary) irreducible if $(n-1 \leqslant)\ p_{1n} \leqslant h$. The *quantum dimension* of such an IR is given by (see, e.g., [18])

$$d_q(p) = \prod_{i=1}^{n-1} \left\{ \frac{1}{[i]!} \prod_{j=i+1}^{n} [p_{ij}] \right\} \qquad (\geqslant 0 \quad \text{for} \quad p_{1n} = p_1 - p_n \leqslant h). \tag{1.42}$$

For $q \to 1$ $(h \to \infty)$, $[m] \to m$ we recover the usual (integral) dimension of the IR under consideration.

The *chiral observable algebra* of the $SU(n)$ WZNW model is generated by a local current $j(x) \in su(n)$ of height h. In contrast to gauge dependent charged fields like $u(x)$, it is periodic, $j(x + 2\pi) = j(x)$. The quantum version of the classical field–current relation $ij(x) = ku'(x)u^{-1}(x)$ is the operator Knizhnik–Zamolodchikov equation [52] in which the level k gets a quantum correction (equal to the dual Coxeter number n of $su(n)$):

$$hu'(x) = i : j(x)u(x) : \qquad h = k + n. \tag{1.43}$$

Here the normal product is defined in terms of the current's frequency parts:

$$ju := j_{(+)}u + uj_{(-)} \qquad j_{(+)}(x) = \sum_{\nu=1}^{\infty} J_{-\nu}\, \mathrm{e}^{\mathrm{i}\nu x} \qquad j_{(-)}(x) = \sum_{\nu=0}^{\infty} J_{\nu}\, \mathrm{e}^{-\mathrm{i}\nu x}. \tag{1.44}$$

The canonical chiral stress energy tensor and the conformal energy L_0 are expressed in terms of j and its modes by the Sugawara formula:

$$\mathcal{T}(x) = \frac{1}{2h}\mathrm{Tr} : j^2 : (x) \Rightarrow L_0 = \int_{-\pi}^{\pi} \mathcal{T}(x)\frac{\mathrm{d}x}{2\pi} = \frac{1}{2h}\mathrm{Tr}\left(J_0^2 + 2\sum_{\nu=1}^{\infty} J_{-\nu}J_{\nu}\right). \tag{1.45}$$

Energy positivity implies that the state space of the chiral quantum WZNW theory is a direct sum of (height h) ground state modules $\mathcal{H}_p$ of the Kac–Moody algebra $\widehat{su}(n)$ each entering with a finite multiplicity:

$$\mathcal{H} = \bigoplus_p \mathcal{H}_p \otimes \mathcal{F}_p \qquad \dim \mathcal{F}_p < \infty. \tag{1.46}$$

We do not fix at this point the structure of the internal spaces $\mathcal{F}_p$. In the simpler but unrealistic case of generic q explored in section 3.1 each $\mathcal{F}_p$ is an irreducible U_q module and the direct sum $\oplus_p \mathcal{F}_p$ carries a Fock-type representation of the intertwining quantum matrix algebra $\mathcal{A}$ introduced below. The irreducibility property fails, in general, for q a root of unity (as discussed in section 3.3). It is conceivable that in this (realistic) case the label p should be substituted by the set of eigenvalues of the U_q Casimir operators which are symmetric polynomials in q^{p_i}. (In the case of $U_q(sl_2)$ the single Casimir invariant depends on $q^p + \bar{q}^p$, $p \equiv p_{12}$, which suggests that p and $2h - p$ should refer to the same internal space.)

Each $\mathcal{H}_p$ in the direct sum (1.46) is a positive energy graded vector space,

$$\mathcal{H}_p = \bigoplus_{\nu=0}^{\infty} \mathcal{H}_p^{\nu} \qquad (L_0 - \Delta(p) - \nu)\,\mathcal{H}_p^{\nu} = 0 \qquad \dim \mathcal{H}_p^{\nu} < \infty. \tag{1.47}$$

It follows from here and from the current algebra and Virasoro CR

$$[J_\nu, L_0] = \nu J_\nu \qquad [L_\nu, L_0] = \nu L_\nu \qquad (\nu \in \mathbb{Z}) \tag{1.48}$$

that $J_\nu \mathcal{H}_p^0 = 0 = L_\nu \mathcal{H}_p^0$ for $\nu = 1, 2, \ldots$. Furthermore, $\mathcal{H}_p^0$ spans an IR of $su(n)$ of (shifted) highest weight p and dimension $d_1(p)$ (the $q \to 1$ limit of the quantum dimension (1.42)). The *conformal dimension* (or *conformal weight*) $\Delta(p)$ is proportional to the ($su(n)$-) second-order Casimir operator $|p|^2 - |\rho|^2$:

$$2h\Delta(p) = |p|^2 - |\rho|^2 = \frac{1}{n} \sum_{1\leqslant i<j\leqslant n} p_{ij}^2 - \frac{n(n^2-1)}{12}. \tag{1.49}$$

Note that the conformal dimension $\Delta(p^{(0)})$ of the trivial representation

$$p^{(0)} = \{p;\ p_{ii+1} = 1,\ 1 \leqslant i \leqslant n-1\} \tag{1.50}$$

is zero. This follows from the identity

$$\begin{aligned} n|p^{(0)}|^2 &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(p_{ij}^{(0)}\right)^2 \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j-i)^2 = \sum_{i=1}^{n-1} \frac{n-i}{6}(2n-2i+1)(n-i+1) \\ &= \frac{n^2(n^2-1)}{12} = n|\rho|^2 \quad \Rightarrow \quad |p^{(0)}|^2 - |\rho|^2 = 0. \end{aligned} \tag{1.51}$$

The eigenvalues of the braid operator $\hat{R}$ (1.9), (1.28) are expressed as products of exponents of conformal dimensions. Let indeed $p^{(1)}$ be the weight of the defining n-dimensional IR of $su(n)$:

$$p_{12}^{(1)} = 2 \qquad p_{ii+1}^{(1)} = 1 \qquad \text{for} \quad i \geqslant 2 \tag{1.52}$$

while $p^{(s)}$ and $p^{(a)}$ are the weights of the symmetric and the antisymmetric squares of $p^{(1)}$, respectively,

$$\begin{aligned} &p_{12}^{(a)} = 1 \quad \left(= p_{ii+1}^{(a)} \text{ for } i \geqslant 3\right) \qquad p_{23}^{(a)} = 2 \quad (\text{for } n \geqslant 3) \\ &p_{12}^{(s)} = 3 \qquad p_{ii+1}^{(s)} = 1 \quad \text{for} \quad n-1 \geqslant i \geqslant 2. \end{aligned} \tag{1.53}$$

The corresponding conformal dimensions $\Delta = \Delta(p^{(1)})$, $\Delta_a = \Delta(p^{(a)})$ and $\Delta_s = \Delta(p^{(s)})$ are computed from (1.49):

$$\begin{aligned} 2h\Delta &= |p^{(1)}|^2 - |\rho|^2 = \frac{n^2-1}{n} \\ 2h\Delta_a &= |p^{(a)}|^2 - |\rho|^2 = 2\frac{n+1}{n}(n-2) \\ 2h\Delta_s &= |p^{(s)}|^2 - |\rho|^2 = 2\frac{n-1}{n}(n+2). \end{aligned} \tag{1.54}$$

The two eigenvalues of $\hat{R}$ (evaluated from the nonvanishing 3-point functions that involve two fields $u(x)$ (or $\varphi(z)$—see (1.12)) of conformal weight Δ,

$$e^{i\pi(2\Delta-\Delta_s)} = q^{\frac{n-1}{n}} \qquad -e^{i\pi(2\Delta-\Delta_a)} = -\bar{q}^{\frac{n+1}{n}} \tag{1.55}$$

appear with multiplicities

$$d_s = \binom{n+1}{2} \qquad d_a = \binom{n}{2} \qquad (d_a + d_s = n^2) \tag{1.56}$$

respectively. The deformation parameter

$$q^{\frac{1}{n}} = e^{-i\frac{\pi}{nh}} \tag{1.57}$$

computed from here, satisfies (0.1) as anticipated. For $q \to 1$ the eigenvalues (1.55) of $\hat{R}$ go into the corresponding eigenvalues ± 1 of the permutation matrix P; furthermore,

$$\det \hat{R} = \det P = (-1)^{d_a} = \begin{cases} -1 & \text{for} \quad n = 2, 3 \bmod 4 \\ 1 & \text{for} \quad n = 0, 1 \bmod 4. \end{cases} \tag{1.58}$$

Equation (1.55) illustrates the early observation (see, e.g., [5]) that the quantum group is determined by basic characteristics (critical exponents) of the underlying conformal field theory.

2. CVO and U_q vertex operators: monodromy and braiding

2.1. Monodromy eigenvalues and $\mathcal{F}_p$ intertwiners

The labels p of the two factors in each term of the expansion (1.46) have different nature. While $\mathcal{H}_p$ is a ground state current algebra module for which p stands for the shifted weight p^{KM} (such that $\big(p_i^{\mathrm{KM}} - p_i\big)\mathcal{H}_p = 0$) of the ground state representation of the $\widehat{su}(n)$ current algebra of minimal conformal dimension (or energy) $\Delta(p)$, $\mathcal{F}_p$ is a U_q module (the quantum group commuting with the currents). We introduce, accordingly, the field of rational functions of the commuting operators $q^{\hat{p}_i}$ (giving rise to an Abelian group) such that

$$\prod_{i=1}^{n} q^{\hat{p}_i} = 1 \qquad (q^{\hat{p}_i} - q^{p_i})\mathcal{F}_p = 0 \qquad [q^{\hat{p}_i}, j(x)] = 0 \tag{2.1}$$

with p_{ij} obeying the condition (1.41) for dominant weights.

We shall split the $SU(n) \times U_q(sl_n)$ covariant field $u(x) = \big(u_\alpha^A(x)\big)$ into factors which intertwine separately different $\mathcal{H}_p$ and $\mathcal{F}_p$ spaces.

A CVO $u_j(x, p)$ (for $p \equiv p^{\mathrm{KM}}$) is defined as an intertwining map between $\mathcal{H}_p$ and $\mathcal{H}_{p+v^{(j)}}$ (for each p in the sum (1.46)). Noting that $\mathcal{H}_p$ is an eigenspace of $\mathrm{e}^{2\pi \mathrm{i} L_0}$,

$$\mathrm{Spec}\, L_0|_{\mathcal{H}_p} \subset \Delta_h(p) + \mathbb{Z}_+ \quad \Rightarrow \quad \{\mathrm{e}^{2\pi\mathrm{i}L_0} - \mathrm{e}^{2\pi\mathrm{i}\Delta_h(p)}\}\mathcal{H}_p = 0 \tag{2.2}$$

we deduce that $u_j(x, p)$ is an eigenvector of the monodromy automorphism,

$$u_j(x + 2\pi, p) = \mathrm{e}^{2\pi\mathrm{i}L_0} u_j(x, p)\, \mathrm{e}^{-2\pi\mathrm{i}L_0} = u_j(x, p)\mu_j(p) \tag{2.3a}$$

where, using (1.49) and the relation $(p|v^{(j)}) = p_j$, we find

$$\mu_j(p) := \mathrm{e}^{2\pi\mathrm{i}\{\Delta_h(p+v^{(j)}) - \Delta_h(p)\}} = q^{\frac{1}{n} - 2p_j - 1}. \tag{2.3b}$$

The monodromy matrix (1.5) is diagonalizable whenever its eigenvalues (2.3b) are all different. In particular, for the 'physical IR', characterized by $p_{1n} < h$, M is diagonalizable. The exceptional points are those p for which there exists a pair of indices $1 \leqslant i < j \leqslant n$ such that $q^{2p_{ij}} = 1$, since we have

$$\frac{\mu_i(p)}{\mu_j(p)} = q^{-2p_{ij}}. \tag{2.3c}$$

According to (1.42) all such 'exceptional' $\mathcal{F}_p$ have zero quantum dimension ($[p_{ij}] = 0$).

Remark 2.1. The simplest example of a non-diagonalizable M appears for $n = 2$, $p\ (\equiv p_{12}) = h$ when $\mu_1(h) = -\bar{q}^{\frac{1}{2}} = \mu_2(h)$. In fact any $\widehat{su}(2)$ module $\mathcal{H}_{h-\ell}$, $0 \leqslant \ell \leqslant h-1$ contains a singular (invariant) subspace isomorphic to $\mathcal{H}_{h+\ell}$ [51]; note that, for $p = h - \ell$, $\tilde{p} = h + \ell$,

$$\mu_1(p) = -q^{\ell - \frac{1}{2}} = \mu_2(\tilde{p}) \qquad \mu_2(p) = -\bar{q}^{\ell + \frac{1}{2}} = \mu_1(\tilde{p}) \tag{2.3d}$$

(cf (2.3b)). It turns out that, in general,

$$\mu_1(p) = \mu_n(\tilde{p}) \qquad \mu_n(p) = \mu_1(\tilde{p}) \qquad \mu_i(p) = \mu_i(\tilde{p}) \qquad i = 2, 3, \ldots, n-1. \tag{2.3e}$$

Indeed, if $|\mathrm{HWV}\rangle_p$ is the highest weight vector in the minimal energy subspace $\mathcal{H}_p^0$ of the $\widehat{su}(n)$ module $\mathcal{H}_p$ and $\theta = \alpha_1 + \cdots + \alpha_{n-1}$ is the $su(n)$ highest root, then the corresponding singular vector can be written in the form [41]

$$\begin{aligned}
&\left(E_{-1}^{\theta}\right)^{h-p_{1n}}|\mathrm{HWV}\rangle_p \sim |\mathrm{HWV}\rangle_{\tilde{p}} \\
&\tilde{p}_1 = h + p_n \qquad \tilde{p}_n = -h + p_1 \qquad \tilde{p}_i = p_i \qquad i = 2, 3, \ldots, n-1 \\
&\Delta(\tilde{p}) - \Delta(p) = h - p_{1n} \in \mathbb{Z}.
\end{aligned} \tag{2.4}$$

To prove (2.3*e*) one only has to insert $\tilde{p}_i$ into (2.3*b*). One concludes that the monodromy M always has coinciding eigenvalues on $\mathcal{F}_p \oplus \mathcal{F}_{\tilde{p}}$ (suggesting the inclusion of this direct sum into a single indecomposable U_q module, cf [32]). The non-diagonalizability of the monodromy matrix in the extended state space may require a modification of the splitting (2.7) below for $p_{1n} \geqslant h$. The study of this question is, however, beyond the scope of the present paper.

The CVO u_j only acts on the factor $\mathcal{H}_p$ of the tensor product $\mathcal{H}_p \otimes \mathcal{F}_p$ in (1.46); hence, it shifts the Kac–Moody operators p_i^{KM} but not the quantum group ones:

$$\left[p_i^{\mathrm{KM}}, u_j\right] = \left(\delta_{ij} - \frac{1}{n}\right) u_j \qquad [q^{\hat{p}_i}, u_j] = 0 \qquad \left(\left(q^{p_i^{\mathrm{KM}}} - q^{\hat{p}_i}\right)\mathcal{H} = 0\right). \tag{2.5}$$

We shall skip from now on (as we did in equations (2.3) and (2.4)) the superscript KM of the argument p of the CVO u_j as well as the hat on the quantum group operators q^{p_i} since the distinction between the two p labels should be obvious from the context (and does not matter when acting on the diagonal chiral state space (1.46)).

The *intertwining quantum matrix algebra* $\mathcal{A}$ is generated by q^{p_i} and by the elements of the matrix $a = \left(a_\alpha^j\right)$, $j, \alpha = 1, \ldots, n$ which shift p (commuting with p^{KM}),

$$a_\alpha^j : \mathcal{F}_p \to \mathcal{F}_{p+v^{(j)}} \qquad q^{p_i} a_\alpha^j = a_\alpha^j q^{p_i + \delta_i^j - \frac{1}{n}}. \tag{2.6}$$

The $SU(n) \times U_q(sl_n)$ covariant field $u_\alpha^A(x)$ is related to the CVO $u_j^A\,(x, p)$ by the so-called vertex-IRF (interaction-round-a-face) transformation [56]

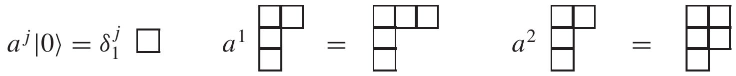
$$u_\alpha^A(x) = u_i^A(x, p) \otimes a_\alpha^i. \tag{2.7}$$

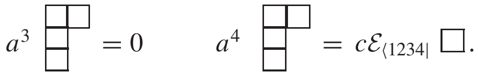
According to (2.7), a_α^j act on the second factor of (1.46) only and hence commute with the currents and with the Virasoro generators.

It will be proved in section 3.1 that the Fock space representation of $\mathcal{A}$ in $\oplus_p \mathcal{F}_p$ for *generic* q provides a model of U_q. The exchange relations of a_α^i (displayed in section 2.3 below), combined with (2.7) and with the defining property $a_\alpha^j|0\rangle = 0$ for $j > 1$ of the vacuum vector $|0\rangle$ (the unique normalized state in $\mathcal{F}_{p^{(0)}}$ for $p^{(0)}$ given by (1.50)) yield, in particular, the relation

$$a^j \mathcal{F}_p = 0 \qquad \text{for} \quad j > 1 \quad \text{and} \quad p_{j-1} = p_j + 1. \tag{2.8}$$

The meaning of (2.6), (2.8) can be visualized as follows. With each finite-dimensional representation of U_q with (dominant) highest weight p we associate, as usual, a *Young tableau* $Y_{[\lambda_1,\ldots,\lambda_{n-1}]}$ with λ_i $(=p_{ii+1} - 1)$ columns of height i $(=1, 2, \ldots, n-1)$. Then a^j adds a box to the jth row of the Young tableau of p (provided $p_{j-1} > p_j + 1$ for $j = 2, \ldots, n$). Here are some examples for $n = 4$:





The exchange relations of a^i_α with the Gauss components (1.5) of the monodromy are dictated by (1.20),

$$M_{1\pm} P a_1 = a_2 \hat{R}^{\mp 1} M_{2\pm} \tag{2.9}$$

and reflect, in view of (1.24a), the U_q covariance of a:

$$\left[E_a, a^i_\alpha\right] = \delta_{a\alpha-1} a^i_{\alpha-1} q^{H_a} \qquad a = 1, \ldots, n-1 \tag{2.10a}$$

$$\left[q^{H_a} F_a, a^i_\alpha\right] = \delta_{a\alpha} q^{H_a} a^i_{\alpha+1} \tag{2.10b}$$

$$q^{H_a} a^i_\alpha = a^i_\alpha q^{H_a + \delta_{a\alpha} - \delta_{a\alpha-1}}. \tag{2.10c}$$

The transformation law (2.10) expresses the coadjoint action of U_q. Comparing (1.2b), (2.3) and (2.7) we deduce that the zero mode matrix a diagonalizes the monodromy (whenever the quantum dimension (1.42) does not vanish); setting

$$aM = M_p a \tag{2.11a}$$

we find (from the above analysis of equation (2.3)) the implication

$$d_q(p) \neq 0 \quad \Rightarrow \quad (M_p)^i_j = \delta^i_j \mu_j(p - v^{(j)}) \qquad \mu_j(p - v^{(j)}) = q^{-2p_j + 1 - \frac{1}{n}}. \tag{2.11b}$$

It follows from (2.11) that the subalgebra of $\mathcal{A}$ generated by the matrix elements of M commutes with all q^{p_i}. As recalled in (1.5), (1.6) and (1.24), the Gauss components of M are expressed in terms of the U_q generators. We can thus state that the centralizer of q^{p_i} in $\mathcal{A}$ is compounded by U_q and q^{p_i}.

2.2. *Exchange relations among zero modes from braiding properties of 4-point blocks*

The exchange relations (1.7a) for u given by (2.7) can be translated into quadratic exchange relations for the 'U_q vertex operators' a^i_α provided that the CVO $u(x, p)$ satisfy standard braid relations: if $0 < x - y < 2\pi$, then

$$\Pi_{xy} u^B_i(y, p + v^{(j)}) u^A_j(x, p) = u^A_k(x, p + v^{(l)}) u^B_l(y, p) \hat{R}(p)^{kl}_{ij}; \tag{2.12}$$

if $0 < y - x < 2\pi$, then $\hat{R}(p)$ on the right-hand side should be substituted by the inverse matrix (cf (1.9b)). Indeed, consistency of (2.12) with (1.7d) on the diagonal state space $\mathcal{H}$ (1.46) requires that

$$\hat{R}(p)^{\pm 1} a_1 a_2 = a_1 a_2 \hat{R}^{\pm 1} \tag{2.13}$$

where p in $\hat{R}(p)$ should be understood as an operator, see (2.1) and (2.6).

It has been proved in [47] that equation (2.12) is in fact a consequence of the properties of the chiral 4-point function

$$w^{(4)}_{p'p} = \langle 0|\phi_{p'^*}(z_1)\varphi(z_2)\varphi(z_3)\phi_p(z_4)|0\rangle \\ = \sum_{i,j} \mathcal{S}^{ij}(p) s_{ij}(z_1, \ldots, z_4; p)\delta_{p', p+v^{(i)}+v^{(j)}} \tag{2.14}$$

(we assume that the vacuum vector is given by the tensor product of the vacuum vectors for the affine and quantum matrix algebras). Here $\phi_p(z)$ and $\phi_{p'^*}(z)$ are general z-picture primary chiral fields of weights p and p'^*, respectively, where p^* is the weight conjugate to p,

$$p \to p^* = \{p^*_i = -p_{n+1-i}\} \quad \Leftrightarrow \quad p^*_{i\,i+1} = p_{n-i\,n+1-i} \tag{2.15}$$

$\varphi(z)$ is the 'step operator' (1.12) (of weight $p^{(1)}$, see (1.52), i.e., $\varphi(z) \equiv \phi_{p^{(1)}}(z)$), $\mathcal{S}^{ij}(p)$ is the zero mode correlator

$$\mathcal{S}^{ij}(p) := \langle p + v^{(i)} + v^{(j)}|a^i a^j|p\rangle \tag{2.16}$$

while s_{ij} is the conformal block expressed in terms of a function of the cross ratio η:

$$s_{ij}\,(z_1, z_2, z_3, z_4; p) := \langle 0|\phi^{p^{(0)}}_{p'^*}(z_1, p')\varphi_i(z_2, p + v^{(j)})\varphi_j(z_3, p)\phi^p_p(z_4, p^{(0)})|0\rangle \\ = D_{ij}(z_1, z_2, z_3, z_4; p) f_{ij}(\eta, p). \tag{2.17}$$

Here we use the standard notation

$$\phi^{p_2}_p(z, p_1) : \mathcal{H}_{p_1} \quad \xrightarrow{\phi_p} \quad \mathcal{H}_{p_2}$$

for a CVO of weight p (so that $\varphi_\ell(z, p) \equiv \phi^{p+v^{(\ell)}}_{p^{(1)}}(z, p)$ is the z-picture counterpart of $u_\ell(x, p)$),

$$D_{ij}\,(z_1, z_2, z_3, z_4; p) = \left(\frac{z_{24}}{z_{12}z_{14}}\right)^{\Delta(p')}\left(\frac{z_{13}}{z_{14}z_{34}}\right)^{\Delta(p)} z_{23}^{-2\Delta}\eta^{\Delta_j-\Delta}(1-\eta)^{\Delta_a}$$

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad p' = p + v^{(i)} + v^{(j)};$$

$\Delta(p)$ is given by (1.49) and $\Delta = \Delta(p^{(1)}) = \frac{n^2-1}{2hn}$, $\Delta_j = \Delta(p + v^{(j)})$, $\Delta_a = \frac{(n+1)(n-2)}{2hn}$ (Δ_a is the dimension (1.54) of the antisymmetric tensor representation of weight $p^{(a)}$ in (1.53)). We omit here both $SU(n)$ and $SL_q(n)$ indices: s_{ij} (2.17) (and hence f_{ij}) is an $SU(n)$ invariant tensor in the tensor product of four IRs, while $\mathcal{S}^{ij}(p)$ (2.16) is an $SL_q(n)$ invariant tensor. Only terms for which both $p + v^{(j)}$ and $p + v^{(i)} + v^{(j)}$ are dominant weights contribute to the sum (2.14). The pre-factor $D_{ij}(z_1, z_2, z_3, z_4; p)$ on the right-hand side of (2.17) is fixed, up to a multiplicative function of η, by the Möbius invariance of f_{ij}. The choice of the powers of η and $1-\eta$ corresponds to extracting the leading singularities (in both s- and u-channels) so that $f_{ij}(\eta, p)$ should be finite (and non-zero) at $\eta = 0$ and $\eta = 1$.

We shall sketch the proof of (2.12); the reader could find the details in [47] (see also [62]).

The conformal block s_{ij} (2.17) is determined as the $SU(n)$ invariant solution of the Knizhnik–Zamolodchikov equation [52]

$$\left(h\frac{\partial}{\partial z_2} + \frac{\mathcal{C}_{12}}{z_{12}} - \frac{\mathcal{C}_{23}}{z_{23}} - \frac{\mathcal{C}_{24}}{z_{24}}\right) s_{ij}(z_1, z_2, z_3, z_4; p) = 0 \tag{2.19a}$$

satisfying the above boundary conditions. Inserting expression (2.17) for s_{ij}, one gets a system of ordinary differential equations for the Möbius invariant amplitudes f_{ij}:

$$\left(h\frac{\mathrm{d}}{\mathrm{d}\eta} - \frac{\Omega_{12}}{\eta} + \frac{\Omega_{23}}{1-\eta}\right) f_{ij}(\eta; p) = 0. \tag{2.19b}$$

Here $\mathcal{C}_{ab} = \vec{t}_a \cdot \vec{t}_b$, $1 \leqslant a < b \leqslant 4$ is the Casimir invariant of the corresponding tensor product of IR of $SU(n)$; in our case t_a, $a = 1, 2, 3, 4$ generate the IR of weights p'^*, $p^{(1)}$, $p^{(1)}$ and p, respectively. The pre-factor D_{ij} being an $SU(n)$ scalar, $SU(n)$ invariance of s_{ij} implies that

$$\left(\mathcal{C}_{12} + \mathcal{C}_{23} + \mathcal{C}_{24} + \frac{n^2-1}{n}\right) f_{ij} = 0 \tag{2.19c}$$

so that

$$\Omega_{12} = \mathcal{C}_{12} + p_m + \delta_{ij} + \frac{n^2+n-4}{2n} \qquad \Omega_{23} = \mathcal{C}_{23} + \frac{n+1}{n} \tag{2.19d}$$

where $m = \min(i, j)$.

Our objective is to study the braiding properties of the solution f_{ij} of (2.19*b*) that is analytic in η (and non-zero) around $\eta = 0$.

It is important to observe that the space of invariant $SU(n)$ tensors is in the case at hand at most two dimensional; this allows us to find a convenient realization of the operators Ω_{12}, Ω_{23} [17, 47]. (In the $n = 2$ case [20, 61, 66] this can be done even for four general isospins, due to the simple rules for tensor multiplication in the $SU(2)$ representation ring.)

The existence of a solution of (2.19) is guaranteed whenever the quantum dimension (1.42) for each weight encountered in (2.14) is positive,

$$n - 1 \leqslant p_{1n}, (p + v^{(j)})_{1n}, {p'}_{1n} < h \qquad p' \equiv p + v^{(i)} + v^{(j)}. \tag{2.20}$$

In fact, for fixed p and p' in (2.17) and $i \neq j$ the 2×2 matrix system (2.19*b*) gives rise to a hypergeometric equation. Assume, in addition, that $p + v^{(i)}$ is also a dominant weight. Then both s_{ij} and s_{ji} will satisfy equation (2.19*a*) and provide a basis of independent solutions of that equation (note that the sum in (2.14) reduces to two terms with permuted i and j). More precisely, let $P_{23}\Pi_{23}s_{ij}(z_1, z_3, z_2, z_4;\, p)$ be the analytic continuation of s_{ij} along a path C_{23} obtained from C_{12} (1.16*a*) by the substitution $1 \to 2, 2 \to 3$ (that is, $C_{23} = \{z_a(t) = \mathrm{e}^{\mathrm{i}\zeta_a(t)}, a = 2, 3;\, \zeta_2(t) + \zeta_3(t) = x_2 + x_3, \zeta_{23}(t) = \mathrm{e}^{-\mathrm{i}\pi t}x_{23}, 0 \leqslant t \leqslant 1\}$) with permuted $SU(n)$ indices 2 and 3. It satisfies again equation (2.19*a*) and hence is a linear combination of $s_{kl}(z_1, z_2, z_3, z_4;\, p)$ with $(k, l) = (i, j)$ and $(k, l) = (j, i)$:

$$P_{23}\Pi_{23}s_{ij}(z_1, z_3, z_2, z_4;\, p) = s_{kl}(z_1, z_2, z_3, z_4;\, p)\hat{R}^{kl}_{ij}(p). \tag{2.21}$$

Here $\hat{R}(p)$ satisfies the *ice condition*: its components $\hat{R}^{kl}_{ij}(p)$ do not vanish only if the *unordered* pairs i, j and k, l coincide, i.e.,

$$\hat{R}^{kl}_{ij}(p) = a^{kl}(p)\delta^k_j\delta^l_i + b^{kl}(p)\delta^k_i\delta^l_j. \tag{2.22a}$$

Equation (2.21) is nothing but a matrix element version of (2.12); hence, it yields the exchange relation (2.13) for $i \neq j$ ($\Rightarrow k \neq l$).

For $i = j$ the analytic continuation on the left-hand side of (2.21) reduces to a multiplication by a phase factor. In this case the space of $SU(n)$ invariant tensors is one dimensional (since the skewsymmetric invariant vanishes), and so is the space of $U_q(s\ell_n)$ invariants. The resulting equation for $f_{ii}(\eta;\, p)$ is of first order:

$$\left(h\frac{\mathrm{d}}{\mathrm{d}\eta} + \frac{2}{1-\eta}\right) f_{ii}(\eta;\, p) = 0$$

and is solved by $f_{ii}(\eta;\, p) = c_{ii}(p)(1-\eta)^{\frac{2}{h}}$. Substituting

$$z_{23} \to \mathrm{e}^{-\mathrm{i}\pi} z_{23} \quad \Rightarrow \quad 1 - \eta \to \mathrm{e}^{-\mathrm{i}\pi}\frac{1-\eta}{\eta} \qquad D_{ii} \to \bar{q}^{\frac{n+1}{n}}\eta^{\frac{2}{h}} D_{ii}$$

we get

$$s_{ii} \overset{\curvearrowright}{\to} q^{1-\frac{1}{n}} s_{ii}.$$

Explicitly, the (4×4) (i, j)-block of $\hat{R}(p)$ has the form

$$\hat{R}^{(i,j)}(p_{ij}) = \bar{q}^{\frac{1}{n}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \frac{q^{p_{ij}}}{[p_{ij}]} & \frac{[p_{ij}-1]}{[p_{ij}]}\alpha(p_{ij}) & 0 \\ 0 & \frac{[p_{ij}+1]}{[p_{ij}]}\alpha(-p_{ij}) & -\frac{\bar{q}^{p_{ij}}}{[p_{ij}]} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \alpha(p)\alpha(-p) = 1 \qquad (2.22b)$$

i.e. (cf (2.22*a*))

$$q^{\frac{1}{n}} a^{kl}(p_{kl}) = \alpha(p_{kl}) \frac{[p_{kl}-1]}{[p_{kl}]} \qquad q^{\frac{1}{n}} b^{kl}(p_{kl}) = \frac{q^{p_{kl}}}{[p_{kl}]} \qquad \text{for} \quad k \neq l.$$

The arbitrariness reflected by $\alpha(p)$ is related to the freedom of choosing the normalization of the two independent solutions of the hypergeometric equation.

The matrix (2.22*b*) coincides with that obtained independently in [45] by imposing consistency conditions on the intertwining quantum matrix algebra of $SL(n)$ type. We shall display the ensuing properties of $\hat{R}(p)$ in the following subsection.

2.3. *The intertwining quantum matrix algebra*

Among the various points of view of the $U_q(sl_2)$ intertwiners (or 'U_q vertex operators') a^i_α, the one which yields an appropriate generalization to $U_q(sl_n)$ is the so-called 'quantum $6j$ symbol' $\hat{R}(p)$-matrix formulation of [2, 15, 30, 56]. The $n^2 \times n^2$ matrix $\hat{R}(p)$ satisfies the *dynamical Yang–Baxter equation* (DYBE) first studied in [42] whose general solution obeying the ice condition was found in [49].

The associativity of triple tensor products of quantum matrices together with equation (1.10) for $\hat{R}$ yields the DYBE for $\hat{R}(p)$:

$$\hat{R}_{12}(p)\hat{R}_{23}(p - v_1)\hat{R}_{12}(p) = \hat{R}_{23}(p - v_1)\hat{R}_{12}(p)\hat{R}_{23}(p - v_1) \qquad (2.23)$$

where we use again the succinct notation of Faddeev *et al* (cf section 1):

$$(\hat{R}_{23}(p - v_1))^{i_1 i_2 i_3}_{j_1 j_2 j_3} = \delta^{i_1}_{j_1} \hat{R}(p - v^{(i_1)})^{i_2 i_3}_{j_2 j_3}. \qquad (2.24)$$

In deriving (2.23) from (2.13) we use (2.6). (The DYBE (2.23) is only *sufficient* for the consistency of the quadratic matrix algebra relations (2.13); it would also be necessary if the matrix a were invertible, i.e., if $d_q(p) \neq 0$.)

The property of the operators $\hat{R}_{i\,i+1}(p)$ to generate a representation of the braid group is ensured by the additional requirement (reflecting (1.27*b*))

$$\hat{R}_{12}(p + v_1 + v_2) = \hat{R}_{12}(p) \quad \Leftrightarrow \quad \hat{R}^{ij}_{kl}(p) a^k_\alpha a^l_\beta = a^k_\alpha a^l_\beta \hat{R}^{ij}_{kl}(p). \qquad (2.25)$$

The Hecke algebra condition (1.27*c*) for the rescaled matrices $\rho_n(g_i)$ (1.28*a*) also fits our analysis of braiding properties of conformal blocks displayed in the previous subsection.

It is not surprising that the direct inspection of the braiding properties of the conformal blocks, on the one hand, and the common solution of the DYBE, (2.25) and the Hecke algebra conditions [45, 49], on the other, lead to the same result. The solution (2.22*b*) can be presented in a form similar to (1.28):

$$q^{\frac{1}{n}} \hat{R}(p) = q \mathbb{1} - A(p) \qquad A(p)^{ij}_{kl} = \frac{[p_{ij}-1]}{[p_{ij}]} \big(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\big). \qquad (2.26)$$

It is straightforward to verify relations (1.29) for $A_i(p) := q\mathbb{1}_{i\,i+1} - q^{\frac{1}{n}} \hat{R}_{i\,i+1}(p)$; in particular,

$$[p_{ij} - 1] + [p_{ij} + 1] = [2][p_{ij}] \quad \Rightarrow \quad A^2(p) = [2]A(p). \qquad (2.27)$$

According to [45] the general $SL(n)$-type dynamical R-matrix [49] can be obtained from (2.26) by either an analogue of Drinfeld's twist [24] (see lemma 3.1 of [45]) or by a canonical transformation $p_i \to p_i + c_i$ where c_i are constants (numbers) such that $\sum_{i=1}^{n} c_i = 0$. The interpretation of the eigenvalues p_i of $\hat{p}_i$ as (shifted) weights (of the corresponding representations of U_q) allows us to dispose of the second freedom.

Inserting (2.26) into the exchange relations (2.13) allows us to present the latter in the following explicit form,

$$\left[a^i_\alpha, a^j_\alpha\right] = 0 \qquad a^i_\alpha a^i_\beta = q^{\epsilon_{\alpha\beta}} a^i_\beta a^i_\alpha \tag{2.28}$$

$$[p_{ij} - 1] a^j_\alpha a^i_\beta = [p_{ij}] a^i_\beta a^j_\alpha - q^{\epsilon_{\beta\alpha} p_{ij}} a^i_\alpha a^j_\beta \qquad \text{for} \quad \alpha \neq \beta \text{ and } i \neq j \tag{2.29}$$

where $q^{\epsilon_{\alpha\beta}}$ is defined in (1.28b).

There is, finally, a relation of order n for a^i_α, derived from the following basic property of the *quantum determinant*,

$$\det(a) = \frac{1}{[n]!} \varepsilon_{\langle 1\ldots n|} a_1 \cdots a_n \mathcal{E}^{|1\ldots n\rangle} \equiv \frac{1}{[n]!} \varepsilon_{i_1\cdots i_n} a^{i_1}_{\alpha_1} \ldots a^{i_n}_{\alpha_n} \mathcal{E}^{\alpha_1\ldots\alpha_n} \tag{2.30}$$

where $\mathcal{E}^{\alpha_1\ldots\alpha_n}$ is given by (1.34) while $\varepsilon_{i_1\ldots i_n}$ is the dynamical Levi-Civita tensor with lower indices (which can be consistently chosen to be equal to the undeformed one [45], a convention which we assume throughout this paper), normalized by $\varepsilon_{n\ldots 1} = 1$. The ratio $\det(a)\left(\prod_{i<j}[p_{ij}]\right)^{-1}$ belongs to the centre of the quantum matrix algebra $\mathcal{A} = \mathcal{A}(\hat{R}(p), \hat{R})$ (see corollary 5.1 of proposition 5.2 in [45]). It is, therefore, legitimate to normalize the quantum determinant setting

$$\det(a) = \prod_{i<j} [p_{ij}] \equiv \mathcal{D}(p). \tag{2.31}$$

It is proportional (with a positive p-independent factor) to the quantum dimension (1.42).

Remark 2.2. The results of this section are clearly applicable if the determinant $\mathcal{D}(p)$ does not vanish (i.e., either for generic q or, if q is given by (0.1), for $p_{1n} < h$). As noted in the introduction, the notion of a CVO and the splitting (2.7) may well require a modification if this condition is violated.

To sum up, *the intertwining quantum matrix algebra $\mathcal{A}$ is generated by the n^2 elements a^i_α and the field $\mathbb{Q}(q, q^{p_i})$ of rational functions of the commuting variables q^{p_i} whose product is 1, subject to the exchange relations (2.6) and (2.13) and the determinant condition (2.31).*

The centralizer of q^{p_i} in $\mathcal{A}$ (i.e., the maximal subalgebra of $\mathcal{A}$ commuting with all q^{p_i}) is spanned by the QUEA U_q over the field $\mathbb{Q}(q, q^{p_i})$ and a^i_α obey the U_q covariance relations (2.10). The expressions for the U_q generators in terms of n-linear products of a^i_α are worked out for $n = 2$ and $n = 3$ in appendix A.

We shall use in what follows the intertwining properties of the product $a_1 \cdots a_n$ (see proposition 5.1 of [45]):

$$\varepsilon_{\langle 1\ldots n|} a_1 \cdots a_n = \mathcal{D}(p) \mathcal{E}_{\langle 1\ldots n|} \tag{2.32a}$$

or, in components,

$$\varepsilon_{i_1\ldots i_n} a^{i_1}_{\alpha_1} \cdots a^{i_n}_{\alpha_n} = \mathcal{D}(p) \mathcal{E}_{\alpha_1\ldots\alpha_n} \tag{2.32b}$$

$$a_1 \cdots a_n \mathcal{E}^{|1\ldots n\rangle} = \varepsilon^{|1\ldots n\rangle}(p) \mathcal{D}(p). \tag{2.33}$$

Here $\varepsilon(p)$ is the dynamical Levi-Civita tensor with upper indices given by

$$\varepsilon^{i_1\dots i_n}(p) = (-1)^{\ell(\sigma)} \prod_{1\leqslant\mu<\nu\leqslant n} \frac{[p_{i_\mu i_\nu} - 1]}{[p_{i_\mu i_\nu}]} \tag{2.34}$$

$\ell(\sigma)$ standing again for the length of the permutation $\sigma = \binom{n,\dots,1}{i_1,\dots,i_n}$.

Remark 2.3. Self-consistency of (1.17) requires that $\det(a) = \det(aM)$. Indeed, the noncommutativity of q^{p_j} and a^i, see equation (2.6), exactly compensates the factors $q^{1-\frac{1}{n}}$ when computing the determinant of aM (cf (2.11*a*), (2.11*b*)); we have

$$q^{2p_n-1+\frac{1}{n}} a^n_{\alpha_1} q^{2p_{n-1}-1+\frac{1}{n}} a^{n-1}_{\alpha_2} \cdots q^{2p_1-1+\frac{1}{n}} a^1_{\alpha_n} = a^n_{\alpha_1} a^{n-1}_{\alpha_2} \cdots a^1_{\alpha_n} \tag{2.35}$$

since

$$q^{\frac{2}{n}(1+2+\dots+n-1)-n+1} = 1. \tag{2.36}$$

An important consequence of the ice property (2.22*a*) (valid for both $\hat{R}$ and $\hat{R}(p)$) is the existence of subalgebras of $\mathcal{A}$ with similar properties.

Let

$$I = \{i_1, i_2, \dots, i_m\} \qquad 1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n$$

and

$$\Gamma = \{\alpha_1, \alpha_2, \dots, \alpha_m\} \qquad 1 \leqslant \alpha_1 < \alpha_2 < \dots < \alpha_m \leqslant n$$

be two ordered sets of m integers ($1 \leqslant m \leqslant n$). Let $A_{1m}|_\Gamma$ be the restriction of the q-antisymmetrizer $(A_{1m})^{\alpha_1\alpha_2\dots\alpha_m}_{\beta_1\beta_2\dots\beta_m}$, $\alpha_k, \beta_k \in \{1, 2, \dots, n\}$ (for its definition see (1.30)) to a subset of indices $\alpha_k, \beta_k \in \Gamma$. Then rank $A_{1m}|_\Gamma = 1$ and one can define the corresponding restricted Levi-Civita tensors satisfying

$$A_{1m}|_\Gamma = \mathcal{E}|_\Gamma{}^{|1\dots m\rangle}\mathcal{E}|_{\Gamma\langle 1\dots m|} \qquad \mathcal{E}|_{\Gamma\langle 1\dots m|}\mathcal{E}|_\Gamma{}^{|1\dots m\rangle} = [m]!. \tag{2.37}$$

In the same way one defines restricted dynamical Levi-Civita tensors

$$\varepsilon|_I{}^{|1\dots m\rangle}(p) \quad \text{and} \quad \varepsilon|_{I\langle 1\dots m|}(p)$$

for the subset $I \subset \{1, 2, \dots, n\}$ (the last one of these does not actually depend on p and coincides with the classical Levi-Civita tensor).

Consider the subalgebra $\mathcal{A}(I, \Gamma) \subset \mathcal{A}$ generated by $\mathbb{Q}(q, q^{p_{ij}})$, $i, j \in I$ and the elements of the submatrix $a|_{I,\Gamma} := \|a\|^{i\in I}_{\alpha\in\Gamma}$ of the quantum matrix a.

Proposition 2.4. *The normalized minor*

$$\Delta_{I,\Gamma}(a) := \frac{\det(a|_{I,\Gamma})}{\mathcal{D}_I(p)} := \frac{1}{[m]!\mathcal{D}_I(p)}\varepsilon|_{I_{\langle 1\dots m|}}(a_1a_2\cdots a_m)|_{I,\Gamma}\mathcal{E}|_\Gamma{}^{|1\dots m\rangle} \tag{2.38}$$

where

$$\mathcal{D}_I(p) := \prod_{i<j;i,j\in I} [p_{ij}] \tag{2.39}$$

belongs to the centre of $\mathcal{A}(I, \Gamma)$.

The statement follows from the observation that relations (2.32*a*)–(2.33) and (2.34) are valid for the restricted quantities $\mathcal{E}|_\Gamma$, $\varepsilon|_I$, $\mathcal{D}_I(p)$ and $a|_{I,\Gamma}$.

Using restricted analogues of relations (2.33) and (2.34), we can derive alternative expressions for the normalized minors,

$$\Delta_{I,\Gamma}(a) = \frac{1}{\mathcal{D}_I^+(p)} a^{i_1}_{\alpha_1} a^{i_2}_{\alpha_2} \cdots a^{i_m}_{\alpha_m} \mathcal{E}\big|_{\Gamma}^{\alpha_1\ldots\alpha_m} \tag{2.40}$$

where

$$\mathcal{D}_I^+(p) := \prod_{i<j;i,j\in I} [p_{ij}+1] \tag{2.41}$$

the indices $i_k \in I$ are in descendant order, $i_1 > i_2 > \cdots > i_m$, and the indices $\alpha_k \in \Gamma$ are summed over.

3. The Fock space representation of $\mathcal{A}$; the ideal $\mathcal{I}_h$ for $q^h = -1$

3.1. The Fock space $\mathcal{F}(\mathcal{A})$ (the case of generic q)

The 'Fock space' representation of the quantum matrix algebra $\mathcal{A}$ was anticipated in equation (2.7) and the subsequent discussion of Young tableaux. We define $\mathcal{F}$ and its dual $\mathcal{F}'$ as cyclic $\mathcal{A}$ modules with one-dimensional (1D) U_q-invariant subspaces of multiples of (non-zero) *bra and ket vacuum vectors* $\langle 0|$ and $|0\rangle$ such that $\langle 0|\mathcal{A} = \mathcal{F}'$, $\mathcal{A}|0\rangle = \mathcal{F}$ satisfying

$$a^i_\alpha|0\rangle = 0 \qquad \text{for} \quad i>1 \qquad \langle 0|a^j_\alpha = 0 \qquad \text{for} \quad j<n \tag{3.1a}$$

$$q^{p_{ij}}|0\rangle = q^{j-i}|0\rangle \qquad \langle 0|q^{p_{ij}} = q^{j-i}\langle 0| \tag{3.1b}$$

$$(X-\varepsilon(X))|0\rangle = 0 = \langle 0|(X-\varepsilon(X)) \tag{3.1c}$$

for any $X \in U_q$ (with $\varepsilon(X)$ the counit). The duality between $\mathcal{F}$ and $\mathcal{F}'$ is established by a bilinear pairing $\langle .|.\rangle$ such that

$$\langle 0|0\rangle = 1 \qquad \langle\Phi|A|\Psi\rangle = \langle\Psi|A'|\Phi\rangle \tag{3.2}$$

where $A \to A'$ is a linear anti-involution (*transposition*) of $\mathcal{A}$ defined for generic q by

$$\mathcal{D}_i(p)\big(a^i_\alpha\big)' = \tilde{a}^\alpha_i := \frac{1}{[n-1]!}\mathcal{E}^{\alpha\alpha_1\ldots\alpha_{n-1}}\varepsilon_{ii_1\ldots i_{n-1}} a^{i_1}_{\alpha_1}\cdots a^{i_{n-1}}_{\alpha_{n-1}} \qquad (q^{p_i})' = q^{p_i} \tag{3.3}$$

where $\mathcal{D}_i(p)$ stands for the product

$$\mathcal{D}_i(p) = \prod_{j<l,\, j\neq i\neq l} [p_{jl}] \quad \big(\Rightarrow \big[\mathcal{D}_i(p), a^i_\alpha\big] = 0 = \big[\mathcal{D}_i(p), \tilde{a}^\alpha_i\big]\big). \tag{3.4}$$

We verify in appendix B the involutivity property, $A'' = A$, of (3.3) for $n=3$. Equation (3.3) implies the following formulae for the transposition of the Chevalley generators of U_q:

$$E_i' = F_i q^{H_i - 1} \qquad F_i' = q^{1-H_i}E_i \qquad (q^{H_i})' = q^{H_i}. \tag{3.5}$$

The main result of this section is the proof of the statement that for generic q (q not a root of unity) $\mathcal{F}$ is a model space for U_q: each finite-dimensional IR of U_q is encountered in $\mathcal{F}$ with multiplicity one.

Lemma 3.1. *For generic q the space $\mathcal{F}$ is spanned by antinormal ordered polynomials applied to the vacuum vector:*

$$P_{m_{n-1}}\big(a^{n-1}_\alpha\big)\cdots P_{m_1}\big(a^1_\alpha\big)|0\rangle \qquad \text{with} \quad m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{n-1}. \tag{3.6}$$

Here $P_{m_i}\big(a^i_\alpha\big)$ is a homogeneous polynomial of degree m_i in $a^i_1,\ldots,a^i_n$.

Proof. We shall first prove the weaker statement that $\mathcal{F}$ is spanned by vectors of the type $P_{m_n}\left(a^n_\alpha\right)\cdots P_{m_1}\left(a^1_\alpha\right)|0\rangle$ (without restrictions on the non-negative integers $m_1,\ldots,m_n$). It follows from the exchange relations (2.29) for $i > j$ and from the observation that $[p_{jl}+1] \neq 0$ for generic q and $j < l$ in view of (3.1*b*).

Next we note that if $m_{j-1} = 0$ but $m_j > 0$ for some $j > 1$, the resulting vector vanishes. Indeed, we can use in this case repeatedly (2.29) for $i < j-1$ to move an a^j_α until it hits the vacuum giving zero according to (3.1*a*).

If all $m_i > 0, i = 1,\ldots,n,$ we move a factor $a^i_{\alpha_i}$ of each monomial to the right to get rid of an n-tuple of $a^i_{\alpha_i}$ since

$$a^n_{\alpha_n}\cdots a^1_{\alpha_1}|0\rangle = [n-1]!\mathcal{E}_{\alpha_n\ldots\alpha_1}|0\rangle; \tag{3.7}$$

here we have used once more (3.1*a*), and also (2.32) and (3.1*b*). Repeating this procedure m_n times, we obtain an expression of the type (3.6) (or zero, if $m_n > \min(m_1,\ldots,m_{n-1})$).

To prove the inequalities $m_i \geqslant m_{i+1}$ we can reduce the problem (by the same procedure of moving, whenever possible, a^i_α to the right) to the statement that any expression of the type $a^{i+1}_{\beta_1}a^{i+1}_{\beta_2}a^i_{\alpha_i}\cdots a^1_{\alpha_1}|0\rangle$ vanishes. We shall display the argument for a special case proving that

$$a^3_\alpha a^3_\beta a^2_2 a^1_1|0\rangle = 0 \qquad \text{for} \quad n \geqslant 3. \tag{3.8}$$

This is a simple consequence of (2.28), (2.29) and (3.1*a*) if either α or β is 1 or 2 . We can hence write, using (2.10*b*),

$$0 = F_2 a^3_2 a^3_3 a^2_2 a^1_1|0\rangle = \left(a^3_3\right)^2 a^2_2 a^1_1|0\rangle + a^3_2 a^3_3 a^2_3 a^1_1|0\rangle = \left(a^3_3\right)^2 a^2_2 a^1_1|0\rangle. \tag{3.9}$$

By repeated application of F_i (with $i \geqslant 3$ for $n \geqslant 4$) exploiting the U_q invariance of the vacuum (3.1*c*), we thus complete the proof of (3.8) and hence, of lemma 3.1. □

Corollary. *It follows from lemma 3.1 that the space $\mathcal{F}$ splits into a direct sum of weight spaces $\mathcal{F}_p$ spanned by vectors of type (3.6) with fixed degrees of homogeneity $m_1,\ldots,m_{n-1}$,*

$$\mathcal{F} = \oplus_p \mathcal{F}_p \qquad p_{ij} = m_i - m_j + j - i \quad (\geqslant j - i \text{ for } i < j) \tag{3.10}$$

each subspace $\mathcal{F}_p$ being characterized by (2.1).

In order to exhibit the U_q properties of $\mathcal{F}_p$ we shall introduce the *highest* and *lowest weight vectors* (HWV and LWV)

$$|\lambda_1\cdots\lambda_{n-1}\rangle \quad \text{and} \quad |-\lambda_{n-1}-\cdots-\lambda_1\rangle$$

obeying

$$(q^{H_i} - q^{\lambda_i})|\lambda_1\cdots\lambda_{n-1}\rangle = 0 = (q^{H_i} - q^{-\lambda_{n-i}})|-\lambda_{n-1}-\cdots-\lambda_1\rangle \tag{3.11}$$

for $\lambda_i = m_i - m_{i+1} = p_{ii+1} - 1, 1 \leqslant i \leqslant n-1$.

Lemma 3.2. *Each $\mathcal{F}_p$ contains a unique (up to normalization) HWV and a unique LWV satisfying (3.11). They can be written in any of the following three equivalent forms:*

$$\begin{aligned}|\lambda_1\cdots\lambda_{n-1}\rangle &= \left(\Delta^{n-1\,1}_{n-1\,1}\right)^{\lambda_{n-1}}\left(\Delta^{n-2\,1}_{n-2\,1}\right)^{\lambda_{n-2}}\cdots\left(\Delta^{2\,1}_{2\,1}\right)^{\lambda_2}\left(a^1_1\right)^{\lambda_1}|0\rangle \\ &= \left(a^1_1\right)^{\lambda_1}\left(\Delta^{2\,1}_{2\,1}\right)^{\lambda_2}\cdots\left(\Delta^{n-1\,1}_{n-1\,1}\right)^{\lambda_{n-1}}|0\rangle \\ &\sim \left(a^{n-1}_{n-1}\right)^{\lambda_{n-1}}\left(a^{n-2}_{n-2}\right)^{\lambda_{n-2}+\lambda_{n-1}}\cdots\left(a^1_1\right)^{\lambda_1+\cdots+\lambda_{n-1}}|0\rangle\end{aligned} \tag{3.12}$$

$$\begin{aligned}|-\lambda_{n-1}-\cdots-\lambda_1\rangle &= \left(\Delta^{n-1\,1}_{n2}\right)^{\lambda_{n-1}}\left(\Delta^{n-2\,1}_{n3}\right)^{\lambda_{n-2}}\cdots\left(\Delta^{2\,1}_{nn-1}\right)^{\lambda_2}\left(a^1_n\right)^{\lambda_1}|0\rangle \\ &= \left(a^1_n\right)^{\lambda_1}\left(\Delta^{2\,1}_{nn-1}\right)^{\lambda_2}\cdots\left(\Delta^{n-1\,1}_{n2}\right)^{\lambda_{n-1}}|0\rangle \\ &\sim \left(a^{n-1}_2\right)^{\lambda_{n-1}}\left(a^{n-2}_3\right)^{\lambda_{n-2}+\lambda_{n-1}}\cdots\left(a^1_n\right)^{\lambda_1+\cdots+\lambda_{n-1}}|0\rangle;\end{aligned} \tag{3.13}$$

here Δ^{i1}_{i1} *and* Δ^{i1}_{nn-i+1} *are normalized minors of the type (2.40),*

$$\Delta^{i1}_{i1} = \Delta_{I_i,\Gamma_i}(a) = \frac{1}{\mathcal{D}^+_{I_i}(p)} a^i_{\alpha_1} \cdots a^1_{\alpha_i} \mathcal{E}\big|^{\alpha_1\ldots\alpha_i}_{\Gamma_i} \tag{3.14}$$

for $I_i := \{1, 2, \ldots, i\} =: \Gamma_i$, *and*

$$\Delta^{i1}_{nn-i+1} \equiv \Delta_{I_i,\Gamma^i_n}(a) = \frac{1}{\mathcal{D}^+_{I_i}(p)} a^i_{\alpha_1} \cdots a^1_{\alpha_i} \mathcal{E}\big|^{\alpha_1\ldots\alpha_i}_{\Gamma^i_n} \tag{3.15}$$

where $\Gamma^i_n := \{n - i + 1, n - i + 2, \ldots, n\}$.

Proof. We shall prove the uniqueness of the HWV by reducing an arbitrary eigenvector of q^{H_i} of eigenvalue q^{λ_i}, $1 \leqslant i \leqslant n-1$, to the form of the second equation (3.12). To this end we again apply the argument in the proof of lemma 3.1. Let k ($\leqslant n-1$) be the maximal numeral for which $\lambda_k > 0$. By repeated application of the exchange relations (2.29) we can arrange each k-tuple $a^k_{\alpha_1} \cdots a^1_{\alpha_k}$ to hit a vector $|v\rangle$ such that $(p_{ii+1} - 1)|v\rangle = 0$ for $i < k$. (Observe that all vectors of the type $|v_k\rangle = \left(\Delta^{k1}_{k1}\right)^{\lambda_k} \cdots \left(\Delta^{n-11}_{n-11}\right)^{\lambda_{n-1}}|0\rangle$, for various choices of the non-negative integers $\lambda_k, \ldots, \lambda_{n-1}$, have this property.) Noting then that $a^{i+1}_\alpha|v\rangle = 0$ whenever $(p_{ii+1} - 1)|v\rangle = 0$ and using once more equation (2.29) we find

$$(p_{ii+1} - 1)|v\rangle = 0 \quad \Leftrightarrow \quad \left(a^{i+1}_\beta a^i_\alpha + q^{\epsilon_{\alpha\beta}} a^{i+1}_\alpha a^i_\beta\right)|v\rangle = 0 \tag{3.16}$$

which implies that we can substitute the product $a^{i+1}_\alpha a^i_\beta$ (acting on such a vector) by its antisymmetrized expression:

$$a^{i+1}_\alpha a^i_\beta |v\rangle = \frac{1}{[2]}\left(q^{\epsilon_{\beta\alpha}} a^{i+1}_\alpha a^i_\beta - a^{i+1}_\beta a^i_\alpha\right)|v\rangle \qquad (\text{for } (p_{ii+1} - 1)|v\rangle = 0). \tag{3.17}$$

Such successive antisymmetrizations will give rise to the minor Δ^{k1}_{k1} yielding eventually the second expression (3.12) for the HWV.

To complete the proof of lemma 3.2, it remains to prove the first equalities in (3.12) and (3.13). The commutativity of all factors Δ^{i1}_{i1}, $1 \leqslant i \leqslant n-1$ $\left(\Delta^{11}_{11} \equiv a^1_1\right)$ follows from proposition 2.4 which implies

$$\left[a^i_\alpha, \Delta^{k1}_{k1}\right] = 0 \qquad \text{for} \quad 1 \leqslant \alpha, i \leqslant k. \tag{3.18}$$

In order to compute the proportionality factors between the second and the third expressions in (3.12) and (3.13) one may use the general exchange relation

$$[p_{ij} - 1]\left(a^j_\alpha\right)^m a^i_\beta = [p_{ij} + m - 1] a^i_\beta \left(a^j_\alpha\right)^m - q^{\epsilon_{\beta\alpha}(p_{ij}+m-1)}[m]\left(a^j_\alpha\right)^{m-1} a^i_\alpha a^j_\beta \tag{3.19}$$

(valid for $i \neq j$ and $\alpha \neq \beta$) which follows from (2.29). □

Lemmas 3.1 and 3.2 yield the main result of this section.

Proposition 3.3. *The space* $\mathcal{F}$ *is (for generic* q*) a model space of* U_q.

We proceed in defining the U_q symmetry of a Young tableau Y. A U_q tensor $T_{\alpha_1\ldots\alpha_s}$ is said to be q-symmetric if for any pair of adjacent indices $\alpha\beta$ we have

$$T_{\ldots\alpha'\beta'\ldots} A^{\alpha'\beta'}_{\alpha\beta} = 0 \quad \Leftrightarrow \quad T_{\ldots\alpha\beta\ldots} = q^{\epsilon_{\alpha\beta}} T_{\ldots\beta\alpha\ldots} \tag{3.20}$$

where $q^{\epsilon_{\alpha\beta}}$ is defined in (1.28*b*). A tensor $F_{\alpha_1\ldots\alpha_s}$ is q-skewsymmetric if it is an eigenvector of the antisymmetrizer (1.28*b*):

$$F_{\ldots\alpha'\beta'\ldots} A^{\alpha'\beta'}_{\alpha\beta} = [2] F_{\ldots\alpha\beta\ldots} \quad \Leftrightarrow \quad F_{\ldots\alpha\beta\ldots} = -q^{\epsilon_{\beta\alpha}} F_{\ldots\beta\alpha\ldots}. \tag{3.21}$$

A U_q tensor of $\lambda_1 + 2\lambda_2 + \cdots + (n-1)\lambda_{n-1}$ indices has the q-symmetry of a Young tableau $Y = Y_{[\lambda_1,\ldots,\lambda_{n-1}]}$ (where λ_i stands for the number of columns of height i) if it is first q-symmetrized in the indices of each row and then q-antisymmetrized along the columns.

The q-symmetry of a tensor associated with a Young tableau allows us to choose as independent components an ordered set of values of the indices α, β that monotonically increase along rows and strictly increase down the columns (as in the undeformed case—see [33]). Counting such labelled tableaux of a fixed type Y allows us to reproduce the dimension $d_1(p)$ of the space $\mathcal{F}_p$.

3.2. Canonical basis; inner product

We shall introduce a *canonical basis* in the U_q modules $\mathcal{F}_p$ in the simplest cases of $n = 2, 3$ preparing the ground for the computation of inner products in $\mathcal{F}_p$ for such low values of n.

We shall follow Lusztig [53] for a general definition of a canonical basis. It is, to begin with, a *basis of weight vectors*, a property which determines it (up to normalization) for $n = 2$. We shall set in this case

$$|p, m\rangle = \left(a_1^1\right)^m \left(a_2^1\right)^{p-1-m} |0\rangle \qquad 0 \leqslant m \leqslant p-1 \quad (p \equiv p_{12}). \tag{3.22}$$

Introducing (following [53]) the operators

$$E^{[m]} = \frac{1}{[m]!} E^m \qquad F^{[m]} = \frac{1}{[m]!} F^m \tag{3.23}$$

we can relate $|p, m\rangle$ to the HWV and LWV in $\mathcal{F}_p$:

$$F^{[p-1-m]} |p, p-1\rangle = \begin{bmatrix} p-1 \\ m \end{bmatrix} |p, m\rangle = E^{[m]} |p, 0\rangle. \tag{3.24}$$

The situation for $n = 3$ can still be handled more or less explicitly. A basis in $\mathcal{F}_p$ is constructed in that case by applying Lusztig's canonical basis [53] in either of the two conjugate Hopf subalgebras of raising or lowering operators

$$X_1^{[m]} X_2^{[\ell]} X_1^{[k]} \qquad \text{and} \qquad X_2^{[k]} X_1^{[\ell]} X_2^{[m]} \qquad \text{for} \quad X = E \text{ or } F \quad \ell \geqslant k + m \tag{3.25}$$

the U_q Serre relations implying

$$X_1^{[m]} X_2^{[k+m]} X_1^{[k]} = X_2^{[k]} X_1^{[k+m]} X_2^{[m]} \tag{3.26}$$

to the lowest or to the highest weight vector, respectively,

$$E_1^{[m]} E_2^{[\ell]} E_1^{[k]} |-\lambda_2 - \lambda_1\rangle \qquad E_2^{[k]} E_1^{[\ell]} E_2^{[m]} |-\lambda_2 - \lambda_1\rangle \tag{3.27}$$

$$F_1^{[m]} F_2^{[\ell]} F_1^{[k]} |\lambda_1 \lambda_2\rangle \qquad F_2^{[k]} F_1^{[\ell]} F_2^{[m]} |\lambda_1 \lambda_2\rangle \qquad 0 \leqslant k + m \leqslant \ell \leqslant \lambda_1 + \lambda_2 \tag{3.28}$$

where we are setting

$$|\lambda_1 \lambda_2\rangle = \left(a_1^1\right)^{\lambda_1} \left(q a_3^{3'}\right)^{\lambda_2} |0\rangle \tag{3.29}$$

$$|-\lambda_2 - \lambda_1\rangle = \left(a_3^1\right)^{\lambda_1} \left(\bar{q} a_1^{3'}\right)^{\lambda_2} |0\rangle. \tag{3.30}$$

(These expressions differ by an overall power of q from (3.12) and (3.13).)

Lemma 3.4. *The action of $F_i^{[m]}$ $\left(E_i^{[m]}\right)$, $m \in \mathbb{N}$ on a HWV (LWV) is given by*

$$\begin{aligned} F_1^{[m]}|\lambda_1\lambda_2\rangle &= \begin{bmatrix}\lambda_1\\ m\end{bmatrix}\left(a_1^1\right)^{\lambda_1-m}\left(a_2^1\right)^m\left(qa_3^{3'}\right)^{\lambda_2}|0\rangle \\ F_2^{[m]}|\lambda_1\lambda_2\rangle &= \begin{bmatrix}\lambda_2\\ m\end{bmatrix}\left(a_1^1\right)^{\lambda_1}\left(qa_3^{3'}\right)^{\lambda_2-m}\left(-a_2^{3'}\right)^m|0\rangle \end{aligned} \tag{3.31a}$$

$$\begin{aligned} E_1^{[m]}|-\lambda_2-\lambda_1\rangle &= \begin{bmatrix}\lambda_2\\ m\end{bmatrix}\left(a_3^1\right)^{\lambda_1}\left(-a_2^{3'}\right)^m\left(\bar{q}a_1^{3'}\right)^{\lambda_2-m}|0\rangle \\ E_2^{[m]}|-\lambda_2-\lambda_1\rangle &= \begin{bmatrix}\lambda_1\\ m\end{bmatrix}\left(a_2^1\right)^m\left(a_3^1\right)^{\lambda_1-m}\left(\bar{q}a_1^{3'}\right)^{\lambda_2}|0\rangle. \end{aligned} \tag{3.31b}$$

The *proof* uses (2.10), (2.28) and the relations

$$a_2^{3'}a_3^{3'} = qa_3^{3'}a_2^{3'} \qquad a_1^{3'}a_2^{3'} = qa_2^{3'}a_1^{3'} \tag{3.32}$$

obtained by transposing the second equality in (2.28) for $i = 3$.

We shall turn now to the computation of the U_q invariant form.

Conjecture 3.5. *The scalar square of the HWV (3.12) and the LWV (3.13) of U_q is given by*

$$\langle\lambda_1\cdots\lambda_{n-1}|\lambda_1\cdots\lambda_{n-1}\rangle = \prod_{i<j}[p_{ij}-1]! = \langle-\lambda_{n-1}-\cdots-\lambda_1|-\lambda_{n-1}-\cdots-\lambda_1\rangle. \tag{3.33}$$

For $n = 2$ the result is a straightforward consequence of equations (3.22) and (A.11) (of appendix A). For $n = 3$ equation (3.33) reads

$$\langle\lambda_1\lambda_2|\lambda_1\lambda_2\rangle = [\lambda_1]![\lambda_2]![\lambda_1+\lambda_2+1]! = \langle-\lambda_2-\lambda_1|-\lambda_2-\lambda_1\rangle \tag{3.34}$$

which is proved in appendix C. We conjecture that the argument can be extended to prove (3.33) for any $n \geqslant 2$.

For $n = 2$ we can also write the inner products of any two vectors of the canonical basis [36]:

$$\langle p, m|p', m'\rangle = \delta_{pp'}\delta_{mm'}\bar{q}^{m(p-1-m)}[m]![p-1-m]!. \tag{3.35}$$

3.3. The case of q a root of unity; subspace of zero norm vectors; ideals in $\mathcal{A}$

In order to extend our results to the study of a WZNW model, we have to describe the structure of the U_q modules $\mathcal{F}_p$ for q a root of unity, (0.1). Here $\mathcal{F}_p$ is, by definition, the space spanned by vectors of type (3.6) (albeit the proof of lemma 3.1 does not apply to this case). We start by recalling the situation for $n = 2$ (see [25, 26, 36]).

The relations

$$E|p, m\rangle = [p-m-1]|p, m+1\rangle \qquad F|p, m\rangle = [m]|p, m-1\rangle \tag{3.36}$$

show that for $p \leqslant h$ the U_q module $\mathcal{F}_p$ admits a single HWV and LWV and is, hence, irreducible. For $p > h$ the situation changes.

Proposition 3.6. *For $h < p < 2h$ and q given by (0.1) the module $\mathcal{F}_p$ is indecomposable. It has two $U_q(sl_2)$ invariant subspaces with no invariant complement:*

$$\begin{aligned} \mathcal{I}_{p,h}^+ &= \mathrm{Span}\{|p, m\rangle, h \leqslant m \leqslant p-1\} \\ \mathcal{I}_{p,h}^- &= \mathrm{Span}\{|p, m\rangle, 0 \leqslant m \leqslant p-1-h\}. \end{aligned} \tag{3.37}$$

It contains a second pair of singular vectors: the LWV $|p, h\rangle$ and the HWV $|p, p-1-h\rangle$. The vector $|p, p-h\rangle$ is cosingular, i.e., it cannot be written in the form $E|v\rangle$ with $v \in \mathcal{F}(p)$; similarly, the vector $|p, h-1\rangle$ cannot be presented as $F|v\rangle$.

The statement follows from (3.36) and from the fact that

$$F|p, p-h\rangle = [p-h]|p, p-h-1\rangle \neq 0 \neq E|p, h-1\rangle \tag{3.38}$$

so that the invariant subspace $\mathcal{I}^+_{p,h} \oplus \mathcal{I}^-_{p,h}$ indeed has no invariant complement in $\mathcal{F}_p$.

The factor space

$$\tilde{\mathcal{F}}_p = \mathcal{F}_p \big/ \left(\mathcal{I}^+_{p,h} \oplus \mathcal{I}^-_{p,h}\right) \qquad (h < p < 2h) \tag{3.39}$$

carries an IR of $U_q(sl_2)$ of weight $\tilde{p} = 2h - p$ (cf (2.4)).

The inner product (3.35) vanishes for vectors of the form (3.22) with $p > h$ and either $m \geqslant h$ or $m \leqslant p - 1 - h$. Writing similar conditions for the bra vectors we end up with the following proposition: *all null vectors belong to the set $\mathcal{I}_h|0\rangle$ or $\langle 0|\mathcal{I}_h$ where $\mathcal{I}_h$ is the ideal generated by $[hp]$, $[hH]$, $q^{hp} + q^{hH}$, and by the hth powers of the a^i_α or, equivalently, by the hth powers of $\tilde{a}^\alpha_i$.* The factor algebra $\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h$ is spanned by monomials of the type

$$q^{\mu p} q^{\nu H} \left(a^1_1\right)^{m_1} \left(a^1_2\right)^{m_2} \left(a^2_1\right)^{n_1} \left(a^2_2\right)^{n_2} \qquad -h < \mu \leqslant h \quad 0 \leqslant \nu < h \quad 0 \leqslant m_i, n_i < h \tag{3.40}$$

and is, hence, (not more than) $2h^6$ dimensional.

The definition of the ideal $\mathcal{I}_h$ can be generalized for any $n \geqslant 2$ assuming that it includes the hth powers of *all minors of the quantum matrix* $\left(a^i_\alpha\right)$ (for $n = 3$, equivalently, the hth powers of a^i_α *and* $\tilde{a}^\alpha_i$). It follows from equation (3.19), taking into account the vanishing of $[h]$, and from (2.28) that

$$\left(a^i_\alpha\right)^h a^j_\beta + (-1)^{\delta_{\alpha\beta}} a^j_\beta \left(a^i_\alpha\right)^h = 0 \quad \left(= \left[[p_{ij}], \left(a^i_\alpha\right)^h\right]_+\right) \tag{3.41}$$

implying also

$$\left(a^i_\alpha\right)^h \tilde{a}^\beta_j + (-1)^{\delta^\beta_\alpha} \tilde{a}^\beta_j \left(a^i_\alpha\right)^h = 0 \qquad \text{for} \quad n = 3. \tag{3.42}$$

Similar relations are obtained (by transposition of (3.41) and (3.42)) for $\left(\tilde{a}^\alpha_i\right)^h$ thus proving that the ideal $\mathcal{I}_h$ is indeed non-trivial, $\mathcal{I}_h \neq \mathcal{A}$.

One can analyse on the basis of lemma 3.4 the structure of indecomposable $U_q(sl_3)$ modules for, say, $h < p_{13} < 3h$, thus extending the result of proposition 3.6. For example, as a corollary of (3.31), for q given by (0.1) (a $2h$th root of 1) a HWV (a LWV) is annihilated by F_i (E_i) if $\lambda_i = 0 \bmod h$ ($\lambda_{\bar{i}} = 0 \bmod h$) where $\lambda_{\bar{1}} = \lambda_2$, $\lambda_{\bar{2}} = \lambda_1$. If, in particular, both λ_i are multiples of h, then the corresponding weight vector spans a 1D IR of $U_q(sl_3)$.

For $n > 2$, however, the subspace $\mathcal{I}_h\,|0\rangle$ does not exhaust the set of null vectors in $\mathcal{F}$. Indeed, for $n = 3$ it follows from (3.34) and from the non-degeneracy of the highest and the lowest weight eigenvalues of the Cartan generators that the HWV and the LWV are null vectors for $p_{13} > h$:

$$\langle \mathcal{F}|\lambda_1\lambda_2\rangle = 0 = \langle \mathcal{F}|-\lambda_2 - \lambda_1\rangle \qquad \text{for} \quad \lambda_1 + \lambda_2 + 1 = p_{13} - 1 \geqslant h. \tag{3.43}$$

(If the conjecture (3.33) is satisfied then the HWV and the LWV for any n are null vectors for $p_{1n} \geqslant h + 1$.) Since the representation of highest weight (λ_1, λ_2) is irreducible for $\lambda_i \leqslant h - 1$ (cf (3.31)), the subspace $\mathcal{N} \subset \mathcal{F}$ of null vectors contains $\mathcal{F}_p$ for $p_{12} = \lambda_1 + 1 \leqslant h$, $p_{23} = \lambda_2 + 1 \leqslant h$, $p_{13} = p_{12} + p_{23} > h$,

$$\mathcal{P}_{\lambda_1\lambda_2}\left(a^1_\alpha; a^{3'}_\beta\right)|0\rangle \in \mathcal{N} \qquad \text{for} \quad \lambda_i \leqslant h - 1 \quad \lambda_1 + \lambda_2 \geqslant h - 1 \tag{3.44}$$

for $\mathcal{P}_{\lambda_1\lambda_2}\left(\rho_1 a^1_\alpha; \rho_2 a^{3'}_\beta\right) = \rho_1^{\lambda_1}\rho_2^{\lambda_2}\mathcal{P}_{\lambda_1\lambda_2}\left(a^1_\alpha; a^{3'}_\beta\right)$, i.e., for any homogeneous polynomial $\mathcal{P}_{\lambda_1\lambda_2}$ of degree λ_1 in the first three variables, a^1_α, and of degree $\lambda_2 \geqslant h - \lambda_1 - 1$ in $a^{3'}_\beta$. It follows that $\mathcal{N}$ contains all U_q modules $\mathcal{F}_{\tilde{p}}$ of weights (2.4) corresponding to the first Kac–Moody singular vector for $p_{13} < h$ ($\Rightarrow \tilde{p}_{13} = 2h - p_{13} > h$, see remark 2.1). Hence, the factor space $\mathcal{F}/\mathcal{N}$ would be too small to accommodate the gauge theory treatment of the zero mode counterpart of such singular vectors.

We can write the null space $\mathcal{N}$ in the form $\mathcal{N} = \tilde{\mathcal{I}}_h|0\rangle$ where $\tilde{\mathcal{I}}_h \subset \mathcal{A}$ is the ideal containing all $\mathcal{P}_{\lambda_1\lambda_2}$ appearing in (3.44) and closed under transposition, which contains $\mathcal{I}_h$ as a proper subideal. (We note that the transposition (3.3) is ill-defined for q a root of unity whenever $\mathcal{D}_i(p)$ vanishes.) The above discussion induces us to define the factor algebra

$$\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h \tag{3.45}$$

(rather than $\mathcal{A}/\tilde{\mathcal{I}}_h$) as the restricted zero mode algebra for q a root of unity. It is easily verified (following the pattern of the $n = 2$ case) that $\mathcal{A}_h$ is again a finite-dimensional algebra. Its Fock space $\mathcal{F}^h$ includes vectors of the form

$$\left(a_1^2\right)^{m_1}\left(a_2^2\right)^{m_2}\left(a_3^2\right)^{m_3}\left(a_1^1\right)^{n_1}\left(a_2^1\right)^{n_2}\left(a_3^1\right)^{n_3}|0\rangle \tag{3.46}$$

for $m_i, n_i < h$ $\left(\sum_i m_i \leqslant \sum_i n_i\right)$ thus allowing for weights

$$p_{13} = n_1 + n_2 + n_3 + 2 \leqslant 3h - 1. \tag{3.47}$$

This justifies the problem of studying indecomposable $U_q(sl_3)$ modules for $p_{13} < 3h$.

To sum up, the intertwining quantum matrix algebra $\mathcal{A}$ introduced in [45] is an appropriate tool for studying the WZNW chiral zero modes. Its Fock space representation provides the first known model of U_q for generic q. For exceptional q (satisfying (0.1)) it gives room—by the results of this section—to the 'physical U_q modules' coupled to the integrable (height h) representations of the $\widehat{su}(n)$ Kac–Moody algebra. This is a prerequisite for a BRS treatment of the zero mode problem of the 2D WZNW model (carried out, for $n = 2$, in [26]).

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Appendix A. Monodromy matrix and identification of $U_q(sl_n)$ generators for $n = 2$ and $n = 3$

Equation (2.11) rewritten as

$$M = a^{-1}M_p a \qquad \text{or} \qquad M^\alpha_\beta = \sum_{i=1}^n (a^{-1})^\alpha_i a^i_\beta q^{-2p_i - 1 + \frac{1}{n}} \tag{A.1}$$

together with the Gauss decomposition (1.5) of the monodromy allows us to express by (1.24a) the Chevalley generators of U_q as well as the operators $E_iE_{i+1} - qE_{i+1}E_i$, $F_{i+1}F_i - qF_iF_{i+1}$ etc as linear combinations of products $a^1_{\alpha_1}\cdots a^n_{\alpha_n}$ (with coefficients that depend on q^{p_i} and the Cartan elements $q^{\pm H_i}$). Indeed, in view of (2.30) and (2.31), we can express the elements of the inverse quantum matrix in terms of the (noncommutative) algebraic complement $\tilde{a}^\alpha_i$ of a^i_α:

$$\mathcal{D}(p)(a^{-1})^\alpha_i = \tilde{a}^\alpha_i = \frac{(-1)^{n-1}}{[n-1]!}\varepsilon_{i_1\ldots i_{n-1}i}\mathcal{E}^{\alpha\alpha_1\ldots\alpha_{n-1}}a^{i_1}_{\alpha_1}\cdots a^{i_{n-1}}_{\alpha_{n-1}}. \tag{A.2}$$

Equation (A.2) is equivalent to (3.3) since for the constant ε-tensor used here we have $(-1)^{n-1}\varepsilon_{i_1\ldots i_{n-1}i} = \varepsilon_{ii_1\ldots i_{n-1}}$. Thus we can recast (2.30)–(2.33) and (A.1) in the form

$$\tilde{a}_i^\alpha a_\beta^i = \mathcal{D}(p)\delta_\beta^\alpha \qquad \sum_{i=1}^n \tilde{a}_i^\alpha a_\beta^i q^{-2p_i-1+\frac{1}{n}} = \mathcal{D}(p)M_\beta^\alpha. \tag{A.3}$$

Using equations (4.10)–(4.12) of [45] we can also write

$$\frac{1}{\mathcal{D}(p)} a_\alpha^i \tilde{a}_j^\alpha = N_j^i(p) = \delta_j^i \prod_{k<i} \frac{[p_{ki}+1]}{[p_{ki}]} \prod_{i<l} \frac{[p_{il}-1]}{[p_{il}]}. \tag{A.4}$$

We can express the U_q generators in terms of products $\tilde{a}_i^\alpha a_\beta^i$ (no summation over i). To this end we use (1.5) and (1.6) to write

$$M_\beta^\alpha = q^{\frac{1}{n}-n} \sum_{\sigma=\max(\alpha,\beta)}^n f_{\alpha\sigma-1} d_\sigma e_{\sigma-1\beta} d_\beta \tag{A.5}$$

with $f_{\alpha\alpha} = f_\alpha, e_{\alpha\alpha} = e_\alpha$; $f_{\alpha\alpha-1} = 1 = e_{\alpha-1\alpha}$ (see (1.24*a*)). It is thus simpler to start the identification of the elements with M_β^n and M_n^α. Using (1.24*a*), we find, in particular,

$$d_n{}^2 = q^{2\Lambda_{n-1}} = \frac{1}{\mathcal{D}(p)} \sum_{i=1}^n \tilde{a}_i^n a_n^i q^{n-1-2p_i}.$$

We shall spell out the full set of resulting relations for $n = 2$ and $n = 3$.

The general relation between Cartan generators and sl_n weights

$$H_i \equiv \sum_j c_{ij}\Lambda_j = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} \qquad (\Lambda_0 = \Lambda_n = 0) \tag{A.6}$$

tells us, for $n = 2$, that $2\Lambda_1 = H$. This allows us (using (1.28*b*) and (1.24*a*)) to write relations (A.3) in the form

$$\tilde{a}_i^\alpha a_\beta^i = [p]\delta_\beta^\alpha \tag{A.7}$$

$$\bar{q}^p \tilde{a}_1^\alpha a_\beta^1 + q^p \tilde{a}_2^\alpha a_\beta^2 = \bar{q}[p](M_+M_-^{-1})_\beta^\alpha. \tag{A.8}$$

Inserting for $M_+M_-^{-1}$ (1.5), (1.6) and (1.24) we find for $n = 2$

$$\begin{aligned} \bar{q}M_+M_-^{-1} &= \bar{q}\begin{pmatrix} \bar{q}^{\frac{H}{2}} & (\bar{q}-q)Fq^{\frac{H}{2}} \\ 0 & q^{\frac{H}{2}} \end{pmatrix}\begin{pmatrix} \bar{q}^{\frac{H}{2}} & 0 \\ (\bar{q}-q)E\bar{q}^{\frac{H}{2}} & q^{\frac{H}{2}} \end{pmatrix} \\ &= \begin{pmatrix} q^p + \bar{q}^p - q^{H+1} & (\bar{q}-q)E' \\ (\bar{q}-q)E & q^{H-1} \end{pmatrix} \qquad E' = Fq^{H-1}. \end{aligned} \tag{A.9}$$

As a result we obtain

$$\begin{aligned} &\bar{q}^p \tilde{a}_1^1 a_1^1 + q^p \tilde{a}_2^1 a_1^2 = [p](q^p + \bar{q}^p - q^{H+1}) \\ &\bar{q}^p \tilde{a}_1^2 a_2^1 + q^p \tilde{a}_2^2 a_2^2 = [p]q^{H-1} \\ &\bar{q}^p \tilde{a}_1^1 a_2^1 + q^p \tilde{a}_2^1 a_2^2 = [p](\bar{q}-q)E' \\ &\bar{q}^p \tilde{a}_1^2 a_1^1 + q^p \tilde{a}_2^2 a_1^2 = [p](\bar{q}-q)E. \end{aligned} \tag{A.10}$$

Together with (A.7) this gives eight equations for the eight products $\tilde{a}_i^\alpha a_\beta^i$ which can be solved with the result

$$\begin{aligned} &\tilde{a}_1^1 a_1^1 = \frac{q^{H+1} - \bar{q}^p}{q - \bar{q}} \quad (= q a_2^2 \tilde{a}_2^2) \qquad \tilde{a}_2^1 a_1^2 = \frac{q^p - q^{H+1}}{q - \bar{q}} \quad (= q a_2^1 \tilde{a}_1^2) \\ &\tilde{a}_2^2 a_2^2 = \frac{q^{H-1} - \bar{q}^p}{q - \bar{q}} \quad (= \bar{q} a_1^1 \tilde{a}_1^1) \qquad \tilde{a}_1^2 a_2^1 = \frac{q^p - q^{H-1}}{q - \bar{q}} \quad (= \bar{q} a_1^2 \tilde{a}_2^1) \\ &\tilde{a}_1^2 a_1^1 = E = -\tilde{a}_2^2 a_1^2 \quad (= a_1^1 \tilde{a}_1^2) \qquad \tilde{a}_1^1 a_2^1 = E' = -\tilde{a}_2^1 a_2^2 \quad (= a_2^1 \tilde{a}_1^1) \end{aligned} \tag{A.11}$$

further implying

$$\begin{aligned} a_2^2\tilde{a}_2^2 - a_1^1\tilde{a}_1^1 &= \bar{q}^p = \bar{q}\tilde{a}_1^1a_1^1 - q\tilde{a}_2^2a_2^2 \\ a_1^2\tilde{a}_2^1 - a_2^1\tilde{a}_1^2 &= q^p = q\tilde{a}_1^2a_2^1 - \bar{q}\tilde{a}_2^1a_1^2 \\ \tilde{a}_1^1a_1^1 - \tilde{a}_2^2a_2^2 &= q^H = \tilde{a}_1^2a_2^1 - \tilde{a}_2^1a_1^2. \end{aligned} \tag{A.12}$$

In deriving the relations including products of the type $a_\alpha^i\tilde{a}_i^\beta$ (appearing in parentheses in (A.11)), we have used (A.2) and (2.28).

These relations agree with (2.28) and (2.29) for $\tilde{a}_i^\alpha$ given by (A.2) which becomes

$$\begin{aligned} &\tilde{a}_i^\alpha = \mathcal{E}^{\alpha\beta}\varepsilon_{ij}a_\beta^j \qquad \text{i.e.} \quad \tilde{a}_1^1 = q^{1/2}a_2^2 \qquad \tilde{a}_2^1 = -q^{1/2}a_2^1 \\ &\tilde{a}_1^2 = -\bar{q}^{1/2}a_1^2 \qquad \tilde{a}_2^2 = \bar{q}^{1/2}a_1^1 \end{aligned} \tag{A.13}$$

implying

$$\tilde{a}_1^1a_1^1 = qa_2^2\tilde{a}_2^2 \qquad \tilde{a}_2^2a_2^2 = \bar{q}a_1^1\tilde{a}_1^1 \qquad \tilde{a}_1^2a_2^1 = \bar{q}a_1^2\tilde{a}_2^1 \qquad \tilde{a}_2^1a_1^2 = qa_2^1\tilde{a}_1^2. \tag{A.14}$$

In the case of $n = 3$ we make (A.3) and (A.4) explicit by noting the identities

$$3p_1 = p_{12} + p_{13} \qquad 3p_2 = p_{23} - p_{12} \qquad 3p_3 = -p_{13} - p_{23} \tag{A.15}$$

$$\mathcal{D}(p) = \mathcal{D}(p_1, p_2, p_3) = [p_{12}][p_{23}][p_{13}] \tag{A.16}$$

$$\begin{aligned} \mathcal{D}(p)N_j^i(p) = \operatorname{diag}(&[p_{23}][p_{12} - 1][p_{13} - 1], [p_{13}][p_{12} + 1][p_{23} - 1], \\ &[p_{12}][p_{13} + 1][p_{23} + 1]) \qquad \left(N_i^i(p) = [3]\right). \end{aligned} \tag{A.17}$$

We find, in particular,

$$\begin{aligned} \mathcal{D}(p)q^{2\Lambda_2-2} &= \tilde{a}_1^3a_3^1\bar{q}^{\frac{2}{3}(p_{12}+p_{13})} + \tilde{a}_2^3a_3^2q^{\frac{2}{3}(p_{12}-p_{23})} + \tilde{a}_3^3a_3^3q^{\frac{2}{3}(p_{13}+p_{23})} \\ \mathcal{D}(p)(\bar{q}^2 - 1)q^{\Lambda_1}E_2 &= \tilde{a}_1^3a_2^1\bar{q}^{\frac{2}{3}(p_{12}+p_{13})} + \tilde{a}_2^3a_2^2q^{\frac{2}{3}(p_{12}-p_{23})} + \tilde{a}_3^3a_2^3q^{\frac{2}{3}(p_{13}+p_{23})} \end{aligned} \tag{A.18}$$

etc.

Appendix B. Transposition in $\mathcal{A}$ for $n = 3$

The involutivity of the transposition (3.3) is easily verified for $n = 2$. Here we shall verify it for $n = 3$ which is indicative for the general case.

Proposition A.1. *The (linear) antihomomorphism of $\mathcal{A}$ defined by (3.3) is involutive:* $a_\alpha^{i\,\prime\prime} = a_\alpha^i$.

Proof. Starting with relation (3.3) for

$$\tilde{a}_i^\alpha = \frac{1}{[2]}\varepsilon_{ijk}\mathcal{E}^{\alpha\beta\gamma}a_\beta^ja_\gamma^k \tag{B.1}$$

we shall prove, say, for $i = 1$, that

$$\begin{aligned} [2][p_{23}]a_\alpha^{1\,\prime\prime} &= \mathcal{E}_{\alpha\beta\gamma}\left(a_\beta^3a_\gamma^2 - a_\beta^2a_\gamma^3\right)' \\ &= \frac{1}{[2]}\mathcal{E}_{\alpha\beta\gamma}\mathcal{E}^{\gamma\rho\sigma}\left\{\frac{1}{[p_{13}]}\left(a_\rho^1a_\sigma^3 - a_\rho^3a_\sigma^1\right)\frac{\tilde{a}_3^\beta}{[p_{12}]} - \frac{1}{[p_{12}]}\left(a_\rho^2a_\sigma^1 - a_\rho^1a_\sigma^2\right)\frac{\tilde{a}_2^\beta}{[p_{13}]}\right\}. \end{aligned} \tag{B.2}$$

Noting the relation between the contraction of two Levi-Civita tensors and the q-antisymmetrizer (1.28b),

$$\mathcal{E}_{\alpha\beta\gamma}\mathcal{E}^{\gamma\rho\sigma} = A_{\alpha\beta}^{\rho\sigma} = \bar{q}^{\epsilon_{\alpha\beta}}\delta_{\alpha\beta}^{\rho\sigma} - \delta_{\beta\alpha}^{\rho\sigma} \tag{B.3}$$

we can rewrite (B.2) as

$$[2]^2\mathcal{D}(p)a^{1''}_\alpha = \frac{[p_{12}]}{[p_{12}-1]}\{\bar{q}^{\epsilon_{\alpha\beta}}(a^1_\alpha a^3_\beta - a^3_\alpha a^1_\beta) - (a^1_\beta a^3_\alpha + a^3_\beta a^1_\alpha)\}\tilde{a}^\beta_3 + \frac{[p_{13}]}{[p_{13}-1]}\{\bar{q}^{\epsilon_{\alpha\beta}}(a^1_\alpha a^2_\beta - a^2_\alpha a^1_\beta) - (a^1_\beta a^2_\alpha + a^2_\beta a^1_\alpha)\}\tilde{a}^\beta_2. \tag{B.4}$$

Applying equation (2.29) four times in the form

$$a^1_\beta a^i_\alpha = \frac{[p_{1i}-1]}{[p_{1i}]}a^i_\alpha a^1_\beta + \frac{\bar{q}^{\epsilon_{\alpha\beta}}}{[p_{1i}]}a^1_\alpha a^i_\beta \qquad a^i_\beta a^1_\alpha = \frac{[p_{1i}+1]}{[p_{1i}]}a^1_\alpha a^i_\beta - \frac{q^{\epsilon_{\alpha\beta}}}{[p_{1i}]}a^i_\alpha a^1_\beta \tag{B.5}$$

for $i = 2, 3$, $\alpha \neq \beta$, and using (A.4), (A.16) and the identities

$$\bar{q}^\epsilon[p] + [p+1] - \bar{q}^{\epsilon p} = [2][p] = \bar{q}^\epsilon[p] + [p-1] + q^{\epsilon p} \tag{B.6}$$

for $\epsilon = \pm 1$, we find that (B.2) is equivalent to

$$[2]\mathcal{D}(p)a^{1''}_\alpha = \frac{[p_{12}]}{[p_{12}-1]}a^1_\alpha[p_{12}][p_{13}+1][p_{23}+1] + \frac{[p_{13}]}{[p_{13}-1]}a^1_\alpha[p_{13}][p_{12}+1][p_{23}-1]$$
$$= [p_{12}][p_{13}]([p_{23}+1] + [p_{23}-1])a^1_\alpha = [2]\mathcal{D}(p)a^1_\alpha. \tag{B.7}$$

The last equality is satisfied due to the CR (2.6) and the 'q-formula'

$$[p-1] + [p+1] = [2][p].$$ □

Appendix C. Computation of the scalar square of highest and lowest weight vectors in the $n = 3$ case

According to the general definition (3.12), the scalar square of the HWV in the $U_q(sl_3)$ module $\mathcal{F}_p$,

$$\langle \mathrm{HWV}(p)|\mathrm{HWV}(p)\rangle = \langle\lambda_1\lambda_2|\lambda_1\lambda_2\rangle \qquad (p_{12} = \lambda_1 + 1,\ p_{23} = \lambda_2 + 1)$$

is given by

$$\langle\lambda_1\lambda_2|\lambda_1\lambda_2\rangle = \langle 0|(qa^3_3)^{\lambda_2}(a^{1'}_1)^{\lambda_1}(a^1_1)^{\lambda_1}(qa^{3'}_3)^{\lambda_2}|0\rangle$$
$$= q^{2\lambda_2}\langle 0|(a^{1'}_1)^{\lambda_1}(a^3_3)^{\lambda_2}(a^{3'}_3)^{\lambda_2}(a^1_1)^{\lambda_1}|0\rangle \tag{C.1}$$

where

$$q[p_{12}+1]a^{3'}_3 = \bar{q}^{\frac{1}{2}}a^2_2 a^1_1 - q^{\frac{1}{2}}a^2_1 a^1_2 \tag{C.2}$$

$$\bar{q}[p_{23}+1]a^{1'}_1 = \bar{q}^{\frac{1}{2}}a^3_3 a^2_2 - q^{\frac{1}{2}}a^3_2 a^2_3. \tag{C.3}$$

We shall prove (3.34) in four steps.

Step 1. *The exchange relation*

$$a^3_3 a^{3'}_3 = \frac{[p_{23}][p_{13}]}{[p_{23}-1][p_{13}-1]}a^{3'}_3 a^3_3 + B_1 a^3_1 + B_2 a^3_2 \tag{C.4}$$

where

$$q^{\frac{3}{2}}[p_{12}+1]B_1 = \bar{q}^{p_{23}}[p_{13}+1]a^2_3 a^1_2 - \bar{q}^{p_{13}}a^2_2 a^1_3$$
$$q^{\frac{1}{2}}[p_{12}+1]B_2 = \bar{q}^{p_{13}}a^2_1 a^1_3 - \bar{q}^{p_{23}}[p_{13}+1]a^2_3 a^1_1 \tag{C.5}$$

obtained by repeated application of (2.29), implies

$$a^3_3(a^{3'}_3)^{\lambda_2}(a^1_1)^{\lambda_1}|0\rangle = \frac{[\lambda_2][\lambda_1+\lambda_2+1]}{[\lambda_1+2]}(a^{3'}_3)^{\lambda_2-1}a^3_3(a^1_1)^{\lambda_1}a^{3'}_3|0\rangle. \tag{C.6}$$

Proof. The last two terms in (C.4), proportional to a_1^3 and a_2^3, do not contribute to (C.6) since, when moved to the right, they yield expressions proportional to $a_\alpha^3 a_3^{3'}|0\rangle$ (=0 for $\alpha = 1, 2$). Repeated application of (C.4) in which only the first term on the right-hand side is kept gives (C.6). □

Step 2. *The exchange relation*

$$[p_{ij} - m]a_\alpha^j \left(a_\beta^i\right)^m = [p_{ij}]\left(a_\beta^i\right)^m a_\alpha^j - q^{\epsilon_{\beta\alpha}(p_{ij}-m+1)}[m]\left(a_\beta^i\right)^{m-1} a_\alpha^i a_\beta^j \tag{C.7}$$

which is a consequence of (2.29), implies

$$a_3^3 \left(a_1^1\right)^{\lambda_1} a_3^{3'}|0\rangle = \frac{[p_{13}]}{[p_{13} - \lambda_1]}\left(a_1^1\right)^{\lambda_1} a_3^3 a_3^{3'}|0\rangle = \bar{q}^2[\lambda_1 + 2]\left(a_1^1\right)^{\lambda_1}|0\rangle. \tag{C.8}$$

Proof. Equation (C.7) is established by induction in m. Equation (C.8) then follows from the identity $q^2 a_3^3 a_3^{3'}|0\rangle = [2]|0\rangle$.

□

Step 3. *Applying λ_2 times steps 1 and 2 one gets*

$$q^{2\lambda_2}\left(a_3^3\right)^{\lambda_2}\left(a_3^{3'}\right)^{\lambda_2}\left(a_1^1\right)^{\lambda_1}|0\rangle = \frac{[\lambda_2]![\lambda_1 + \lambda_2 + 1]!}{[\lambda_1 + 1]!}\left(a_1^1\right)^{\lambda_1}|0\rangle. \tag{C.9}$$

Step 4. *Equations (3.19), (C.7) and (2.29) imply,*

$$\left(a_1^{1'}\right)\left(a_1^1\right)^{\lambda_1}|0\rangle = [\lambda_1][\lambda_1 + 1]\left(a_1^1\right)^{\lambda_1 - 1}|0\rangle; \tag{C.10}$$

as a result,

$$\langle\lambda_1 0|\lambda_1 0\rangle = \langle 0|\left(a_1^{1'}\right)^{\lambda_1}\left(a_1^1\right)^{\lambda_1}|0\rangle = [\lambda_1]![\lambda_1 + 1]!. \tag{C.11}$$

The last two steps are obvious.

An analogous computation gives the same result (3.34) for the scalar square of the LWV $\langle -\lambda_2 - \lambda_1| -\lambda_2 - \lambda_1\rangle$.

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